Initial Algebra Semantics for Cyclic Sharing Structures

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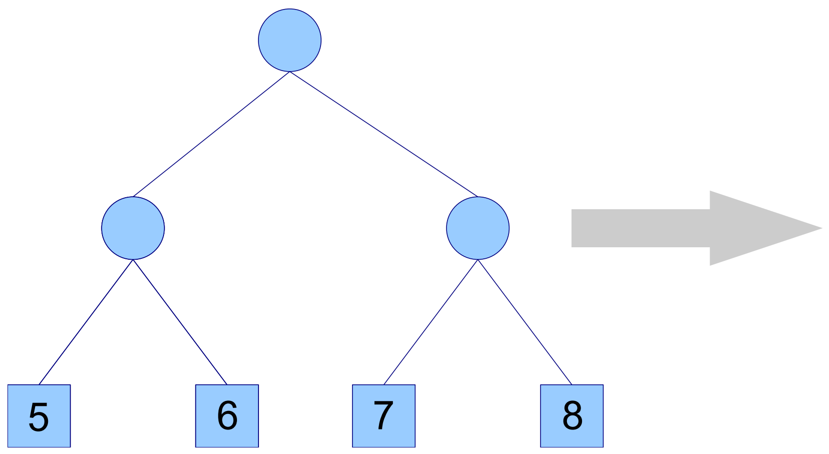
This Work

- ▷ How to inductively capture cylces and sharing
- ▷ Intended to apply it to functional programming

Introduction

▷ Terms are a representation of tree structures



bin(bin(5,6),bin(7,8))

(i) Reasoning: structural induction

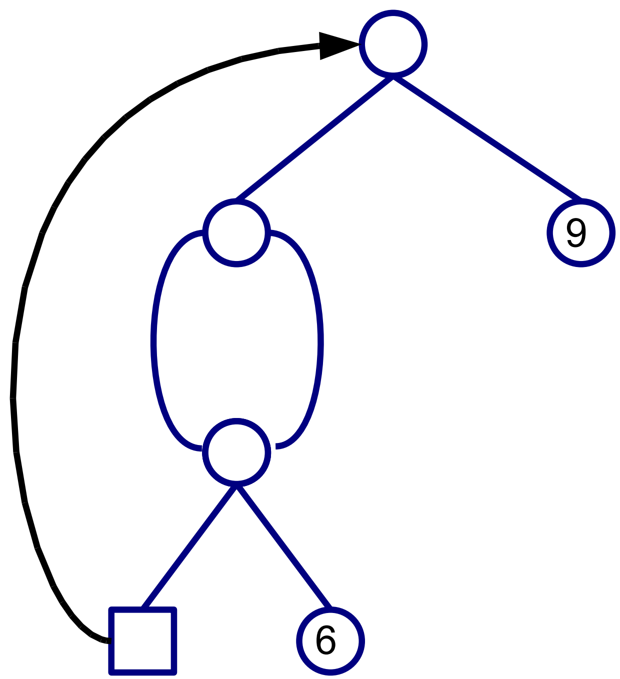
(ii) Functional programming:
pattern matching, structural recursion

(iii) Type: inductive type

(iv) Initial algebra property

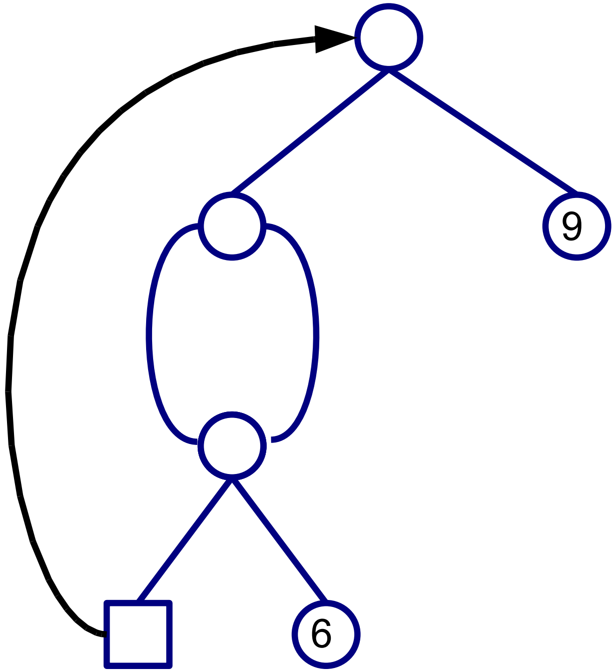
Introduction

- What about tree-like structures?



- How can we represent this data in functional programming?
- Maybe: vertices and edges set, adjacency lists, etc.
- Give up to use pattern matching, structural induction
- Not inductive

Introduction



Are really no inductive structures in tree-like structures?

This Work

- ▷ Gives an initial algebra characterisation of cyclic sharing structures

Aim

- ▷ To derive the following from ↑ :

[I] A simple term syntax that admits structural induction / recursion

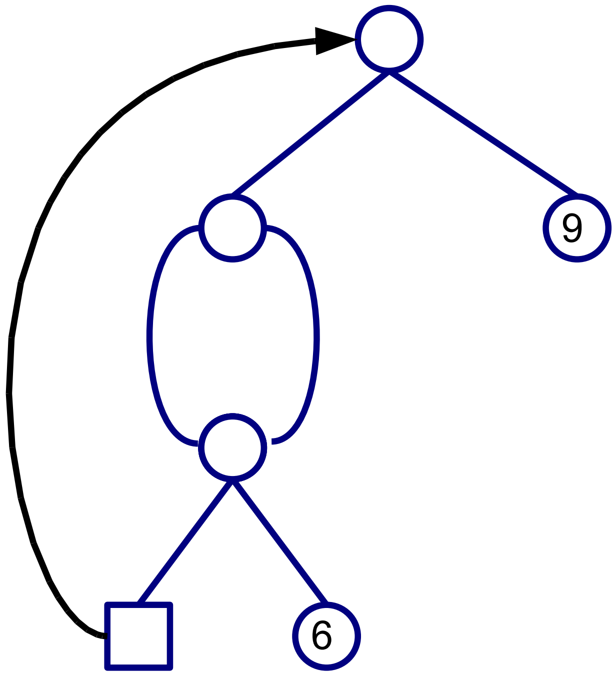
[II] To give an inductive type that represents cyclic sharing structures uniquely in functional languages and proof assistants

Variations of Initial Algebra Semantics

▷ Various computational structures are formulated as initial algebras by varying the base category

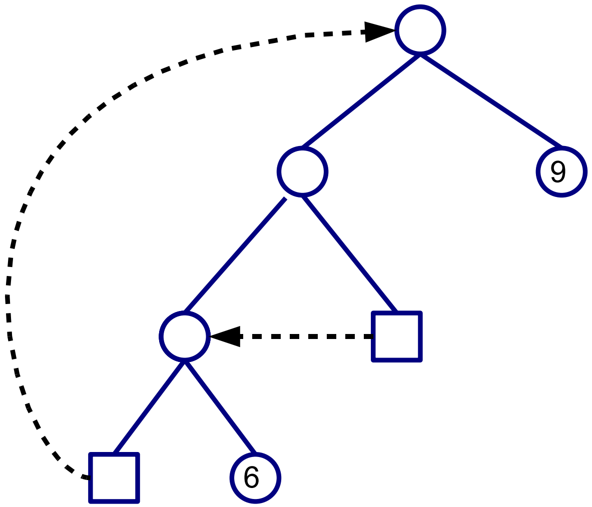
Abstract syntax	$\mathbf{Set}$	ADJ	1975
S-sorted abstract syntax	$\mathbf{Set}^S$	Robinson	1994
Abstract syntax with binding	$\mathbf{Set}^{\mathbb{F}}$	Fiore, Plotkin, Turi	1999
Recursive path ordring	$\mathbf{LO}$	R. Hasegawa	2002
S-sorted 2nd-order abs. syn.	$(\mathbf{Set}^{\mathbb{F}\downarrow S})^S$	Fiore	2003
2nd-order rewriting systems	$\mathbf{Pre}^{\mathbb{F}}$	Hamana	2005
Explicit substitutions	$[\mathbf{Set}, \mathbf{Set}]_f$	Ghani, Uustalu, Hamana	2006
Cyclic sharing structures	$(\mathbf{Set}^{\mathbb{T}^*})^{\mathbb{T}}$	Hamana	2009

Basic Idea



Basic Idea: Graph Algorithmic View

- Traverse a graph in a depth-first search manner:



Depth-First Search tree

- DFS tree consists of 3 kinds of edges:

(i) Tree edge (ii) Back edge

(iii) Right-to-left cross edge

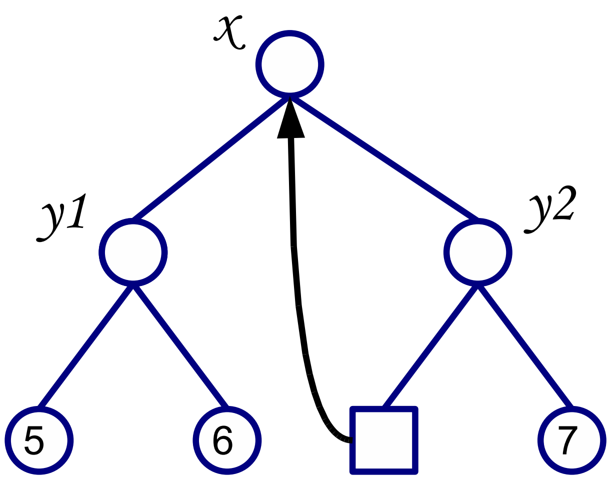
- Characterise pointers for back and cross edges

Formulation: Cycles by μ-terms

Idea

- *Binders* as *pointers*
- Back edges = bound variables

Cycles



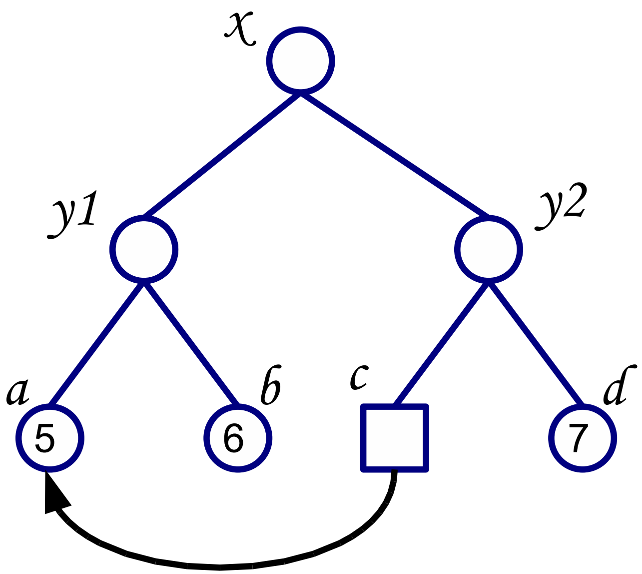
$$\mu x.\mathbf{bin}(\mu y_1.\mathbf{bin}(\mathbf{lf}(5), \mathbf{lf}(6)), \ \mu y_2.\mathbf{bin}(\ x, \ \mathbf{lf}(7)\ ))$$

Formulation: Sharing via ?

Idea

- *Binders as pointers*
- Back edges = bound variables
- Right-to-left Cross edges = ?

Sharing

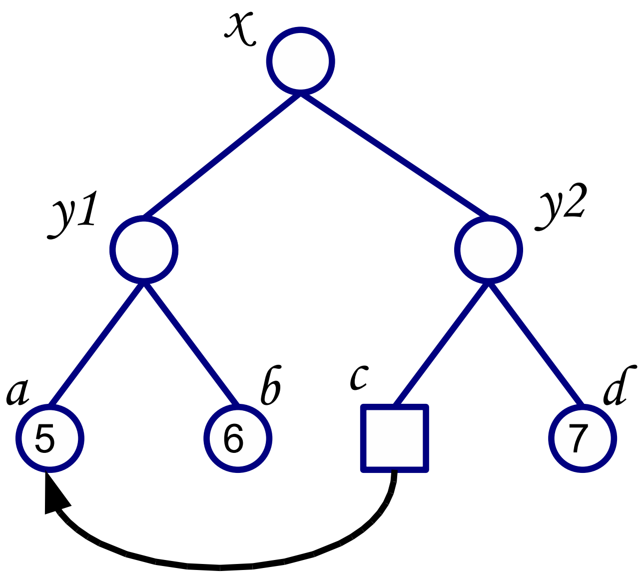


$$\mu x.\mathbf{bin}(\mu y_1.\mathbf{bin}(\mathbf{lf}(5), \mathbf{lf}(6)),\ \mu y_2.\mathbf{bin}(\boxed{}, \mathbf{lf}(7))).$$

Can we fill the blank to refer the node 5 by a bound variable?

Formulation: Sharing via Pointer

- ▷ Cross edges = pointers   by a new notation



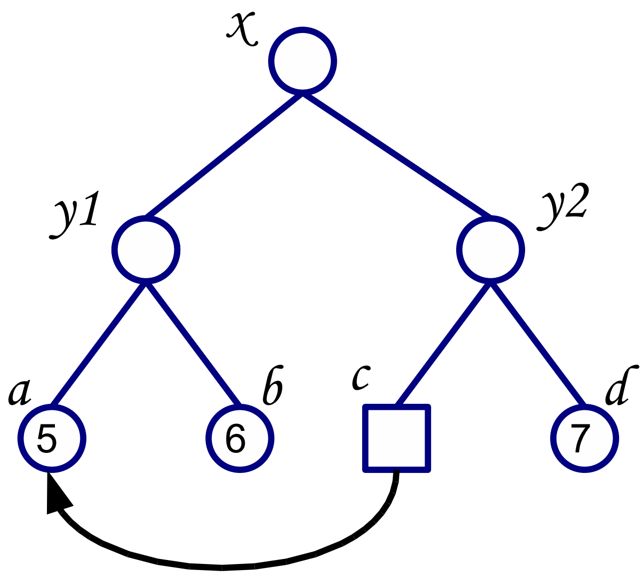
$$\mu x.\mathbf{bin}(\mu y_1.\mathbf{bin}(\mathbf{lf}(5), \mathbf{lf}(6)),\ \mu y_2.\mathbf{bin}(\boxed{\swarrow 11 \uparrow x}, \mathbf{lf}(7)))$$

Pointer $\swarrow 11 \uparrow x$ means

- ▷ going back to the node x, then
- ▷ going down through the left child twice (by position **11**)

Formulation: Sharing via Pointer

▷ Cross edges = pointers by a new notation



$$\mu x.\mathbf{bin}(\mu y_1.\mathbf{bin}(\mathbf{lf}(5), \mathbf{lf}(6)), \ \mu y_2.\mathbf{bin}(\boxed{\swarrow 11 \uparrow x}, \mathbf{lf}(7)))$$

Pointer $\swarrow 11 \uparrow x$ means Need to ensure a correct pointer only!!

▷ going back to the node x, then

▷ going down through the left child twice (by position $\mathbf{11}$)

Typed Abstract Syntax

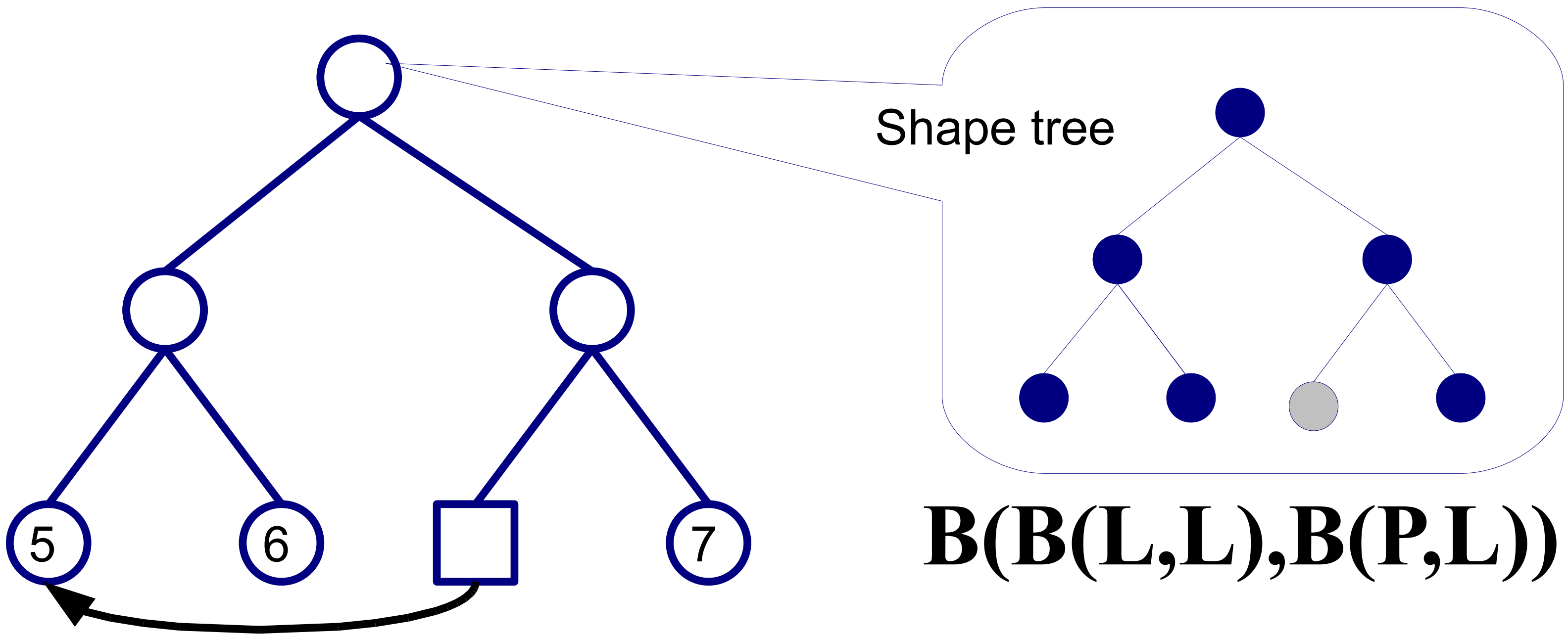
for

Cyclic Sharing Structures

Shape Trees

- ▷ Skeltons of cyclic sharing trees

$$\text{Shape trees} \quad \tau ::= \mathbf{E} \quad | \quad \mathbf{P} \quad | \quad \mathbf{L} \quad | \quad \mathbf{B}(\tau_1, \tau_2)$$



- ▷ Used as types
- ▷ Blue nodes represent possible positions for sharing pointers.

Syntax and Types

Typing rules

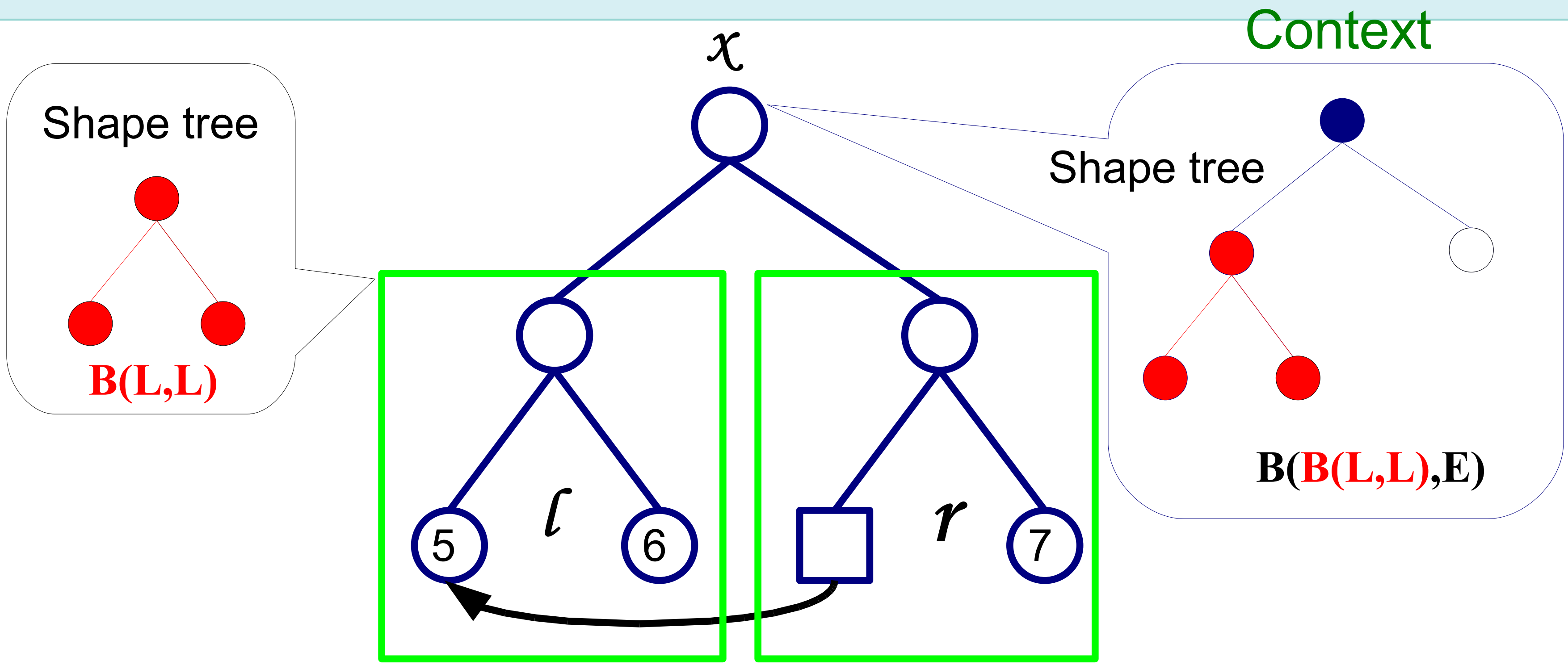
$$\frac{p \in \mathcal{P}os(\sigma)}{\Gamma, x : \sigma, \Gamma' \vdash \swarrow p{\uparrow}x : \mathrm{P}} \qquad \frac{k \in \mathbb{Z}}{\Gamma \vdash \mathsf{lf}(k) : \mathrm{L}}$$

$$\frac{x : \mathrm{B}(\mathrm{E}, \mathrm{E}), \Gamma \vdash \ell : \sigma \qquad x : \mathrm{B}(\sigma, \mathrm{E}), \Gamma \vdash r : \tau}{\Gamma \vdash \mu x.\mathsf{bin}(\ell, r) : \mathrm{B}(\sigma, \tau)}$$

- A type declaration $x : \sigma$ means:
  "σ is the shape of a subtree headed by μx".

- Taking a position $p \in \mathcal{P}os(\sigma)$ safely refers to a position in the subtree.

Example: making bin-node



$$\frac{x{:}\mathbf{B}(\mathbf{E},\mathbf{E}) \vdash \mu y_1.\mathsf{bin}(5,6) : \mathbf{B}(\mathbf{L},\mathbf{L}) \qquad x{:}\mathbf{B}(\mathbf{B}(\mathbf{L},\mathbf{L}),\mathbf{E}) \vdash \mu y_2.\mathsf{bin}(\swarrow 11\uparrow x, 7) : \mathbf{B}(\mathbf{P},\mathbf{L})}{\vdash \mu x.\mathsf{bin}(\mu y_1.\mathsf{bin}(5,6), \mu y_2.\mathsf{bin}(\swarrow 11\uparrow x, 7)) : \mathbf{B}(\mathbf{B}(\mathbf{L},\mathbf{L}),\mathbf{B}(\mathbf{P},\mathbf{L}))}$$

Syntax and Types

Typing rules (de Bruijn version)

$$\frac{|\Gamma| = i - 1 \quad p \in \mathcal{P}os(\sigma)}{\Gamma, \sigma, \Gamma' \vdash \swarrow p{\uparrow}i : \mathrm{P}} \qquad \frac{k \in \mathbb{Z}}{\Gamma \vdash \mathsf{lf}(k) : \mathrm{L}}$$

$$\frac{\mathrm{B}(\mathrm{E},\mathrm{E}), \Gamma \vdash s : \sigma \quad \mathrm{B}(\sigma,\mathrm{E}), \Gamma \vdash t : \tau}{\Gamma \vdash \mathsf{bin}(s,t) : \mathrm{B}(\sigma,\tau)}$$

Thm.

Given rooted, connected and edge-ordered graph, the term representation in de Bruijn is unique.

Initial Algebra Semantics

▷ Cyclic sharing trees are all well-typed terms:

$$T_{\tau}(\Gamma) = \{t \mid \Gamma \vdash t : \tau\}$$

▷ Need: sets indexed by
contexts $\mathbb{T}^*$ and shape trees $\mathbb{T}$

Consider algebras in $(\mathbf{Set}^{\mathbb{T}^*})^{\mathbb{T}}$

Initial Algebra Semantics

- ▷ $\boldsymbol{\Sigma}$-algebra $(\boldsymbol{A}, \boldsymbol{\alpha} : \boldsymbol{\Sigma A} \rightarrow \boldsymbol{A})$

- ▷ Functor $\boldsymbol{\Sigma} : (\mathbf{Set}^{\mathbb{T}^*})^{\mathbb{T}} \longrightarrow (\mathbf{Set}^{\mathbb{T}^*})^{\mathbb{T}}$ for cyclic sharing trees is defined by

$$(\boldsymbol{\Sigma A})_{\mathrm{E}} = \mathbf{0} \qquad (\boldsymbol{\Sigma A})_{\mathrm{P}} = \mathbf{PO} \qquad (\boldsymbol{\Sigma A})_{\mathrm{L}} = \boldsymbol{K}_{\mathbb{Z}}$$

$$(\boldsymbol{\Sigma A})_{\mathrm{B}(\sigma,\tau)} = \boldsymbol{\delta}_{\mathrm{B}(\mathrm{E},\mathrm{E})} \boldsymbol{A}_{\sigma} \times \boldsymbol{\delta}_{\mathrm{B}(\sigma,\mathrm{E})} \boldsymbol{A}_{\tau}$$

Initial Algebra Semantics

- Σ-algebra $(A,\ \alpha : \Sigma A \to A)$
- Functor $\Sigma : (\mathbf{Set}^{\mathbb{T}^*})^{\mathbb{T}} \longrightarrow (\mathbf{Set}^{\mathbb{T}^*})^{\mathbb{T}}$ for cyclic sharing trees is given by

$$\mathsf{ptr}^A : \mathrm{PO} \to A_{\mathrm{P}} \qquad \mathsf{lf}^A : K_{\mathbb{Z}} \to A_{\mathrm{L}}$$

$$\mathsf{bin}^{\sigma,\tau\ A} : \delta_{\mathrm{B(E,E)}} A_\sigma \times \delta_{\mathrm{B}(\sigma,\mathrm{E})} A_\tau \to A_{\mathrm{B}(\sigma,\tau)}$$

Typing rules (de Bruijn version)

$$\frac{|\Gamma| = i-1 \quad p \in \mathcal{P}os(\sigma)}{\Gamma, \sigma, \Gamma' \vdash \swarrow p\uparrow i : \mathrm{P}} \qquad \frac{k \in \mathbb{Z}}{\Gamma \vdash \mathsf{lf}(k) : \mathrm{L}}$$

$$\frac{\mathrm{B(E,E)}, \Gamma \vdash s : \sigma \quad \mathrm{B}(\sigma, \mathrm{E}), \Gamma \vdash t : \tau}{\Gamma \vdash \mathsf{bin}(s,t) : \mathrm{B}(\sigma,\tau)}$$

Initial Algebra

▷ All cyclic sharing trees

$$\boldsymbol{T}_{\boldsymbol{\tau}}(\boldsymbol{\Gamma}) = \{\boldsymbol{t} \mid \boldsymbol{\Gamma} \vdash \boldsymbol{t} : \boldsymbol{\tau}\}$$

Thm. $\boldsymbol{T}$ forms an initial $\boldsymbol{\Sigma}$-algebra.

[Proof]

▷ Smith-Plotkin construction of an initial algebra

Principles

The initial algebra characterisation derives

(i) Structural recursion by the unique homomorphism

(ii) Structural induction by [Hermida,Jacobs I&C'98]

(iii) Inductive type (in Haskell)

Inductive Type for Cyclic Sharing Structures

Constructors of the initial algebra $T \in (\mathbf{Set}^{\mathbb{T}^*})^{\mathbb{T}}$

$$\mathbf{ptr}^T(\Gamma) : \mathrm{PO}(\Gamma) \longrightarrow T_{\mathrm{P}}(\Gamma); \quad \swarrow p{\uparrow}i \mapsto \swarrow p{\uparrow}i.$$

$$\mathbf{lf}^T(\Gamma) : \mathbb{Z} \longrightarrow T_{\mathrm{L}}(\Gamma); \quad k \mapsto \mathbf{lf}(k).$$

$$\mathbf{bin}^{\sigma,\tau\,T}(\Gamma) : T_\sigma(\mathrm{B}(\mathrm{E},\mathrm{E}),\Gamma) \times T_\tau(\mathrm{B}(\sigma,\mathrm{E}),\Gamma) \longrightarrow T_{\mathrm{B}(\sigma,\tau)}(\Gamma)$$

GADT in Haskell

```
data T :: * -> * -> * where
  Ptr :: Ctx n => n -> T n P
  Lf  :: Ctx n => Int -> T n L
  Bin :: (Ctx n, Shape s, Shape t) =>
         T (TyCtx (B E E) n) s -> T (TyCtx (B s E) n) t
                                    -> T n (B s t)
```

▷ Dependent type def. in Agda is more straightforward

Summary

▷ An initial algebra characterisation

Goals

▷ To derive the following from ↑ :

[I] A simple term syntax

[II] An inductive type

for cyclic sharing structures

Connections to Other Works

There are interpretations:

$$\boldsymbol{T} \xrightarrow{\;!\;} \text{Equational Term Graphs} \longrightarrow \boldsymbol{\mathcal{S}}$$

where $\boldsymbol{\mathcal{S}}$ is any of

(i) Coalgebraic

(ii) Domain-theoretic

(iii) Categorical semantics:
Traced sym. monoidal categories [M. Hasegawa TLCA'97]

–

(Equational) term graphs [Barendregt et al.'87][Ariola,Klop'96]

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Further applications

(cyclic) $\boldsymbol{T} \longrightarrow \boldsymbol{\mathcal{C}}$ (cartesian-center traced)

$\sim$ "arrows" with loops in Haskell

$\boldsymbol{T} \longrightarrow$ **Haskell** (efficient implementation)