Universal Algebra for Termination of Higher-Order Rewriting

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Abstract. We show that the structures of binding algebras and Σ-monoids by Fiore, Plotkin and Turi are sound and complete models of Klop's Combinatory Reduction Systems (CRSs). These algebraic structures play the same role of universal algebra for term rewriting systems. Restricting the algebraic structures to the ones equipped with well-founded relations, we obtain a complete characterisation of terminating CRSs. We can also naturally extend the characterisation to rewriting on meta-terms by using the notion of Σ-monoids.

1 Introduction

At RTA'98, Plotkin presented the theory of *binding algebras* [Plo98], which aimed to apply ideas in universal algebra to type theory. It is interesting that this was given as an invited talk at RTA. That is to say, in the context of rewriting, it can be read as a possibility of a new direction of foundation of *higher-order rewriting* as a type theoretic system. Plotkin's idea of binding algebras was inspired by Aczel's work [Acz78]. In the field of rewriting, also inspired by Aczel's same work, Klop invented a system of higher-order rewriting called *Combinatory Reduction System* (CRS) [Klo80]. It is natural to think that these two works, having a common origin, have some relationship. However, such a relationship is not obvious, especially about how the seemingly complex syntax of CRSs can be understood in the theory of binding algebras.

Plotkin's program of binding algebras later produced the notion of Σ-monoids [FPT99]. Interestingly, the *free* Σ*-monoids* constructed in [Ham04] is the same as the syntax of "meta-terms" of CRSs (cf. Theorem 5). This similarity suggests that the universal algebra for CRSs may be Σ-monoids. Based on this idea, the present paper provides a complete algebraic characterisation of CRSs.

Contribution. Complete characterisation of terminating CRSs obtained in this paper provides a method of proving the termination of CRSs by algebraic interpretation. The following CRS $\mathcal{R}$ for conversion into prenex normal form, i.e. pushing quantifiers outside, is a typical example of higher-order rewrite rules that require the feature of variable binding [Pol96, Raa]:

$$\mathrm{P} \wedge \forall(x.\mathrm{Q}[x]) \to \forall(x.\mathrm{P} \wedge \mathrm{Q}[x]) \qquad \neg\forall(x.\mathrm{Q}[x]) \to \exists(x.\neg(\mathrm{Q}[x]))$$
$$\forall(x.\mathrm{Q}[x]) \wedge \mathrm{P} \to \forall(x.\mathrm{P} \wedge \mathrm{Q}[x]) \qquad \neg\exists(x.\mathrm{Q}[x]) \to \forall(x.\neg(\mathrm{Q}[x]))$$

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with the similar rewrite rules for $\vee$ and $\exists$ at the left column. Intuitively, rewriting using $\mathcal{R}$ and its termination are clear; notwithstanding, the application of existing proof methods in the theory of higher-order rewriting to the CRS $\mathcal{R}$ is not so straightforward [JR01], or it requires consideration of an involved function space to interpret binders [Pol96, Pol94]. The present paper provides a simpler method of showing termination of CRS such as $\mathcal{R}$ (cf. Example 24).

Organisation. This paper is organised as follows. We first review the definition of CRSs in Section 2. We then introduce the notion of "structural CRSs". define a class of structurally well-formed CRSs in Section 3. Section 4 gives algebraic semantics of CRSs syntax and valuations. Section 5 gives algebraic semantics of CRSs rewriting. Section 6 gives algebraic semantics of CRSs meta-rewriting. Finally, in Section 7, we show examples of termination proofs using a result of this paper.

Future work. This work opens a new direction of model theoretic study of higher-order rewriting. An immediate application will be semantic labelling method [Zan94] for CRSs using the algebraic structure developed in this paper. Recursive path ordering on free structures in more general setting is also hopeful.

2 Combinatory Reduction Systems

We review the definition of CRSs. We use the definition of the standard reference [KOR93] of CRSs with a slight modification of syntax used in [DR98]: $-.-$ and $-[-]$ instead of ordinary ones $[-]-$ and $-(-)$ in [KOR93].

CRS. Assume a signature Σ of function symbols F^l with arity, metavariables z^l with arity (in both cases the superscript $l \in \mathbb{N}$ is the arity).

(i) CRS *terms* have the form

$$t ::= x \mid x.t \mid F^l(t_1, \ldots, t_l).$$

The three forms are respectively called *variables*, *abstractions*, and *function terms*.

(ii) CRS *meta-terms* extend CRS terms to

$$t ::= x \mid x.t \mid F^l(t_1, \ldots, t_l) \mid \mathrm{z}^l\,[t_1, \ldots, t_l].$$

The last form is called a *meta-application*.

(iii) A *valuation* θ is a mapping that assigns to n-ary metavariable z an n-ary *substitute* (a meta-level lambda notation, cf. [KOR93]):

$$\theta : \mathrm{z} \longmapsto \underline{\lambda}(x_1, \ldots, x_n).t \tag{1}$$

Valuations are extended to a function on meta-terms:

$$\theta(x) = x \qquad \theta(F(t_1, \ldots, t_l)) = F(\theta(t_1), \ldots, \theta(t_l))$$
$$\theta(x.t) = x.\theta(t) \qquad \theta(\mathrm{z}[t_1, \ldots, t_l]) = \theta(\mathrm{z})\,(\theta(t_1), \ldots, \theta(t_l)) \tag{2}$$

Note that the rhs of the equation (2) uses an application at the meta-level to the substitute. The valuation is *safe* if there are no two substitutes $\theta(\mathrm{Z})$ and $\theta(\mathrm{Z'})$ such that $\theta(\mathrm{Z})$ contains a free variable x which appears also bound in $\theta(\mathrm{Z'})$.

(iv) CRS *rules*, written $l \to r$, consist of two meta-terms l and r with the following additional restrictions:
  (iv-a) l and r are closed (w.r.t. variables) meta-terms,
  (iv-b) l must be a "pattern", i.e. a function term where all meta-applications have the form $\mathrm{Z}[x_1, \ldots, x_n]$ with distinct x_i,
  (iv-c) r can only contain meta-applications with meta-variables occurring in the left-hand side.

  The rewrite rule $l \to r$ is *safe for θ,* if for no Z in l and r, the substitute $\theta(\mathrm{Z})$ has a free variable x occurring in an abstraction $x.-$ of l and r. A set of rewrite rules is called a CRS.

(v) The CRS *rewrite relation* $\to_{\mathcal{R}}$ is generated by context and safe valuation closure of a given CRS $\mathcal{R}$:

$$\frac{l \to r \in \mathcal{R}}{\theta(l) \to_{\mathcal{R}} \theta(l)}\ \text{safe } \theta \qquad \frac{s \to_{\mathcal{R}} t}{x.s \to_{\mathcal{R}} x.t} \qquad \frac{s \to_{\mathcal{R}} t}{F(\ldots, s, \ldots) \to_{\mathcal{R}} F(\ldots, t, \ldots)}$$

  where $l \to r$ must be safe for the safe valuation θ. The third rule means a rewriting at the i-th argument of F.

3 Structural CRSs

In this section, we introduce the notion of *structural* CRS as a class of well-formed CRSs. This idea of structural CRS is to treat only CRS (meta-)terms built from binding signature (cf. Aczel's contraction schemes [Acz78]). A binding signature specifies how many binders are taken in arguments of each function symbol.

Formally, a *binding signature* Σ is consisting of a set Σ of function symbols with an arity function $a : \Sigma \to \mathbb{N}^*$. A function symbol of *binding arity* $\langle n_1, \ldots, n_l \rangle$, denoted by $f : \langle n_1, \ldots, n_l \rangle$, has l arguments and binds n_i variables in the i-th argument ($1 \leq i \leq l$).

For a formal treatment of named variables modulo α-equivalence in CRSs, we assume the method of de Bruijn levels [dB72, LRD95, FPT99] for the naming convention of variables (N.B. not for metavariables) in CRSs. We also use the convention that $n \in \mathbb{N}$ denotes the set $\{1, \ldots, n\}$ (n is possibly 0). Under the method of de Bruijn levels, this n means the set of variables from 1 to n.

Definition 1. A (meta-)term t is called *structural* if t is built from a binding signature Σ and consistent with the binding arities of function symbols in Σ.

Schematically, structural meta-terms have the form:

$$t ::= x \mid F(x_1 \cdots x_{i_1}.t_1\,, \ldots,\, x_1 \cdots x_{i_l}.t_l) \mid \mathrm{Z}^l\,[t_1, \ldots, t_l]$$

where F has the binding arity $\langle i_1, \ldots, i_l \rangle$.

More precisely, structural meta-terms are defined as follows. Fix an $\mathbb{N}$-indexed set Z of metavariables defined by $Z(l) \triangleq \{\mathrm{z} \mid \mathrm{z} \text{ has arity } l\}$. A meta-term t is structural if $n \vdash t$ is derived from the following rules.

$$\frac{x \in n}{n \vdash x} \qquad \frac{F : \langle i_1, \ldots, i_l \rangle \in \Sigma \quad n{+}i_1 \vdash t_1 \ \cdots \ n{+}i_l \vdash t_l}{n \vdash F(\, n{+}1 \ldots n{+}i_1.t_1,\ \ldots,\ n{+}1 \ldots n{+}i_l.t_l\,)}$$

$$\frac{\mathrm{z} \in Z(l) \quad n \vdash t_1 \ \cdots \ n \vdash t_l}{n \vdash \mathrm{z}[t_1, \ldots, t_l]}$$

By these rules, a meta-term always follows the method of de Bruijn levels. Using only the first two rules (or equivalently, assuming $Z(l) = \varnothing$ for all l), we obtain *structural terms* under n.

The notion of structural is obviously extended to rewrite rules, CRS, and valuations. A rewrite rule is called structural if all meta-terms in the rule are structural. A CRS is structural if all rules are structural.

Definition 2. A valuation θ is *structural* if for any mapping by $\theta : \mathrm{z} \mapsto \underline{\lambda}(x_1, \ldots, x_n).t$, t is a structural term and all variables in t are included in $x_1, \ldots, x_n$.

Structual CRS is a fairly good assumption because we can easily find that almost all concrete examples of CRSs considered in the literature are structural; namely, we can easily find a suitable binding signature of a given "plain" CRS. Actually, in Raamsdonk's collection [Raa] of examples of higher-order rewrite systems *all CRSs are structural*.

Example 3 (CPS translation). The format of structural CRS is very similar to an "everyday" meta-language for expressing formal systems in computer science and logic. An example is the structural CRS $\mathcal{R}$ for prenex normal form in the introduction. Another example related to theory of programming languages is the following CRS $\mathcal{S}$ of a call-by-value CPS translation [DR98].

Assume the metavariables $Z = \{\mathrm{V}^0, \mathrm{E}^1, (\mathrm{E}_0)^0, (\mathrm{E}_1)^0\}$ and the binding signature Σ consisting of the function symbols $\lambda, \overline{\lambda} : \langle 1 \rangle,\ (-\ -), (-\,{}^{-}\,-) : \langle 0, 0 \rangle$, $\mathsf{CPS}, (\![-]\!) : \langle 0 \rangle$. We write the structural CRS $\mathcal{S}$ of CPS translation in two ways: the left column is written in the usual named notation, and the right column is written in de Bruijn level notation, which is the format we use in this paper.

$\mathsf{CPS}(\mathrm{E}) \to \lambda k.(\![\mathrm{E}]\!)\,{}^{-}\,(\overline{\lambda} m.km)$	$\mathsf{CPS}(\mathrm{E}) \to \lambda 1.(\![\mathrm{E}]\!)\,{}^{-}\,(\overline{\lambda} 2.12)$
$(\![\mathrm{V}]\!) \to \overline{\lambda} k.k\,{}^{-}\,\mathrm{V}$	$(\![\mathrm{V}]\!) \to \overline{\lambda} 1.1\,{}^{-}\,\mathrm{V}$
$(\![\lambda x.\mathrm{E}[x]]\!) \to \overline{\lambda} k.k\,{}^{-}\,(\lambda x.\lambda k.(\![\mathrm{E}[x]]\!)\,{}^{-}\,(\overline{\lambda} m.km))$	$(\![\lambda 1.\mathrm{E}[1]]\!) \to \overline{\lambda} 1.1\,{}^{-}\,(\lambda 2.\lambda 3.(\![\mathrm{E}[2]]\!)\,{}^{-}\,(\overline{\lambda} 4.34))$
$(\![\mathrm{E}_0 \mathrm{E}_1]\!) \to \overline{\lambda} k.\ (\![\mathrm{E}_0]\!)\,{}^{-}\,(\overline{\lambda} m.(\![\mathrm{E}_1]\!)\,{}^{-}\,(\overline{\lambda} n.mn(\lambda a.k\,{}^{-}\,a)))$	$(\![\mathrm{E}_0 \mathrm{E}_1]\!) \to \overline{\lambda} 1.\ (\![\mathrm{E}_0]\!)\,{}^{-}\,(\overline{\lambda} 2.(\![\mathrm{E}_1]\!)\,{}^{-}\,(\overline{\lambda} 3.23(\lambda 4.1\,{}^{-}\,4)))$

A point is that de Bruijn level version is obtained by just renaming variable names with numbers according to their (de Bruijn's) levels. Notice that this completely differs from the more well-known method of de Bruijn *indexes*. Meta-terms in de Bruijn levels are just "normal forms" of α-equivalent meta-terms (e.g. $\overline{\lambda} k.k\,{}^{-}\,\mathrm{V} =_\alpha \overline{\lambda} 1.1\,{}^{-}\,\mathrm{V}$).

Is the structural CRS $\mathcal{S}$ terminating[1]? Intuitively, termination is clear because $(\!|-|\!)$ recursively decomposes a λ-term. In this paper, we derive a formal way of showing termination from an algebraic characterisation of rewriting of CRS. How this $\mathcal{S}$ is shown to be terminating will be given in Example 25 at the end of the paper.

4 Algebraic Semantics of Syntax

In this section and in the next section, we consider algebraic semantics of CRSs. As far as the author knows, this is the first algebraic consideration of CRSs. The basic idea is similar to the algebraic semantics of TRSs by monotone Σ-algebras popularized by Zantema [Zan94]. But the framework of usual first-order universal algebra is insufficient. We consider CRS's syntax in the framework of *binding algebras* by Fiore, Plotkin and Turi [FPT99].

4.1 Binding Algebras

We review the notion of binding algebras. For detail, see [FPT99]. Let $\mathbb{F}$ be the category which has finite cardinals $n = \{1, \ldots, n\}$ (n is possibly 0) as objects, and all functions between them as arrows $m \to n$. This is the category of object variables by the method of de Bruijn levels (i.e. natural numbers) and their renamings. We use the functor category $\mathbf{Set}^{\mathbb{F}}$. We define the functor $\delta : \mathbf{Set}^{\mathbb{F}} \to \mathbf{Set}^{\mathbb{F}}$ as follows: for $L \in \mathbf{Set}^{\mathbb{F}}, n \in \mathbb{F}, \rho \in \operatorname{arr} \mathbb{F}$, $(\delta L)(n) = L(n+1)$, $\quad (\delta L)(\rho) = L(\rho + \mathrm{id}_1)$. To a binding signature Σ, we associate the *signature functor* $\Sigma : \mathbf{Set}^{\mathbb{F}} \to \mathbf{Set}^{\mathbb{F}}$ given by $\Sigma A \triangleq \coprod_{f:\langle n_1,\ldots,n_l\rangle \in \Sigma} \prod_{1 \le i \le l} \delta^{n_i} A$. A Σ-*binding algebra* (or simply Σ-*algebra*) is a pair (A, α) consisting of a presheaf $A \in \mathbf{Set}^{\mathbb{F}}$ and a map ([ ] denotes a copair of coproducts) $\alpha = [f_A]_{f \in \Sigma} : \Sigma A \longrightarrow A$ called *algebra structure*, where f_A is an *operation* $f_A : \delta^{n_1} A \times \ldots \times \delta^{n_l} A \longrightarrow A$ defined for each function symbol $f : \langle n_1, \ldots, n_l \rangle \in \Sigma$.

The "the presheaf of variables" $\mathrm{V} \in \mathbf{Set}^{\mathbb{F}}$ is defined by $\mathrm{V}(n) = n$, $\mathrm{V}(\rho) = \rho \quad (\rho : m \to n \in \mathbb{F})$. Then, $(\mathbf{Set}^{\mathbb{F}}, \bullet, \mathrm{V})$ forms a monoidal category [Mac71], where the "substitution" monoidal product is defined as follows. For presheaves A and B, $(A \bullet B)(n) \triangleq (\coprod_{m \in \mathbb{N}} A(m) \times B(n)^m)/\sim$ where $\sim$ is the equivalence relation generated by $(t; u_{\rho 1}, \ldots, u_{\rho m}) \sim (A(\rho)(t); u_1, \ldots, u_l)$ for $\rho : m \to l \in \mathbb{F}$. Throughout the paper, we use the following notation: an element of $A(m) \times B(n)^m$ is denoted by $(t; u_1, \ldots, u_m)$ where $t \in A(m)$ and $u_1, \ldots, u_m \in B(m)$. A representative of an equivalence class in $A \bullet B(n)$ is also denoted by this notation.

Let Σ be a signature functor with strength *st* defined by a binding signature. A Σ-*monoid* $M = (\alpha, \eta, \mu)$ consists of a *monoid object* [Mac71] $(M, \eta : \mathrm{V} \to M, \mu : M \bullet M \to M)$ in the monoidal category $(\mathbf{Set}^{\mathbb{N}}, \bullet, \mathrm{V})$ with a Σ-binding algebra $\alpha : \Sigma M \to M$ satisfying $\mu \circ (\alpha \bullet \mathrm{id}_M) = \alpha \circ \Sigma\mu \circ st$. A Σ-*monoid morphism* $(M, \alpha) \longrightarrow (M', \alpha')$ is a morphism in $\mathbf{Set}^{\mathbb{F}}$ which is both Σ-algebra homomorphism and monoid morphism.

[1] This does not contain β-reduction rules, i.e. only for translation.

4.2 Algebra of Structural CRS Terms

Structural terms and meta-terms have a good algebraic structure. We define the presheaf $T_\Sigma \mathrm{V} \in \mathbf{Set}^{\mathbb{F}}$ of all structural terms under n by $T_\Sigma\mathrm{V}(n) = \{t \mid n \vdash t,\ t \text{ is a term}\}$ with obvious arrow part [Ham04]. We also define the map $\nu : \mathrm{V} \longrightarrow T_\Sigma\mathrm{V}$ in $\mathbf{Set}^{\mathbb{F}}$ by $\nu(n) : \mathrm{V}(n) \longrightarrow T_\Sigma\mathrm{V}(n),\ x \longmapsto x$. We abbreviate $n{+}1, \ldots, n{+}k.t$ to $n{+}\vec{k}.t$. For every $f \in \Sigma$ with the arity $\langle i_1, \ldots, i_l\rangle$, we define the map $F_T : \delta^{i_1} T_\Sigma\mathrm{V} \times \cdots \times \delta^{i_l} T_\Sigma\mathrm{V} \longrightarrow T_\Sigma\mathrm{V}$ in $\mathbf{Set}^{\mathbb{F}}$ by $(t_1, \ldots, t_l) \longmapsto F(n{+}\vec{i_1}.t_1, \ldots, n{+}\vec{i_l}.t_l)$.

Theorem 4. *Structural CRS terms $T_\Sigma\mathrm{V}$ forms an initial $\mathrm{V}+\Sigma$-binding algebra.*

Proof. Due to [FPT99]. The "syntactic algebra" in ([FPT99] Theorem 2.1) is nothing but the $\mathrm{V} + \Sigma$-algebra $(T_\Sigma\mathrm{V}, [\nu, [F_{T_\Sigma}]_{F\in\Sigma}])$. □

Moreover, let Z be an arbitrary $\mathbb{N}$-indexed set of metavariables (cf. Sect. 3). The *presheaf $M_\Sigma Z$ of meta-terms* is defined by

$$M_\Sigma Z(n) = \{t \mid n \vdash t\}.$$

There is the map $\beta : M_\Sigma Z \bullet M_\Sigma Z \longrightarrow M_\Sigma Z$ in $\mathbf{Set}^{\mathbb{F}}$, called *multiplication*, that performs a substitution for variables [Ham04].

Theorem 5. *Structural CRS meta-terms $M_\Sigma Z$ forms a free Σ-monoid over $\hat{Z}$, where $\hat{Z}(n) = \coprod_{k\in\mathbb{N}} \mathbb{F}(k, n) \times Z(k)$.*

Proof. Due to [Ham04]. For $\hat{Z} \in \mathbf{Set}^{\mathbb{N}}$, the free Σ-monoid constructed in [Ham04] is nothing but $(M_\Sigma Z, [F_{M_\Sigma}]_{F\in\Sigma}, \nu, \beta)$ by just identifying minor notational difference of terms: regard $\mathsf{ovar}(x),\ [n]t,\ \lceil \mathrm{z} \rceil \langle t_1, \ldots, t_l\rangle$ in [Ham04] as $x,\ n.t,\ \mathrm{z}[t_1, \ldots, t_l]$ respectively in the present paper. Here, operations F_{M_Σ} are defined by the same as F_{T_Σ}. □

4.3 Algebraic Characterisation of Valuations

Definition 6. An *assignment* $\phi : Z \longrightarrow A$ is a morphism of $\mathbf{Set}^{\mathbb{N}}$ whose target A has a Σ-monoid structure (A, ν, β).

Notice that Z in the above definitions is a presheaf in $\mathbf{Set}^{\mathbb{F}}$. So just an $\mathbb{N}$-indexed set X of metavariables cannot be the source of this presentation of valuation. Fortunately, we can always construct a presheaf from an $\mathbb{N}$-indexed set X by defining $\hat{X}(n) \triangleq \coprod_{k\in\mathbb{N}} \mathbb{F}(k, n) \times X(k)$ (see [Ham04] Sect. 5.2). Hence, hereafter we abuse the notation to use X to denote its presheaf version $\hat{X} \in \mathbf{Set}^{\mathbb{F}}$ in an assignment.

An assignment ϕ is extended to a Σ-monoid morphism $\phi^* : M_\Sigma Z \longrightarrow A$:

$$
\begin{aligned}
M_\Sigma Z(n) &\longrightarrow A(n) \\
x &\longmapsto \nu(n)(x) \qquad (x \in n) \\
F(n{+}\vec{i_1}.t_1, \ldots, n{+}\vec{i_l}.t_l) &\longmapsto F_A(n{+}\vec{i_1}.\phi^*(n{+}i_1)(t_1), \ldots, n{+}\vec{i_l}.\phi^*(n{+}i_l)(t_l)) \\
\mathrm{z}[t_1, \ldots, t_l] &\longmapsto \beta(n)(\, \phi(l)(\mathrm{z});\ \phi^*(n)(t_1), \ldots \phi^*(n)(t_l)\,)
\end{aligned}
$$

where $f : \langle i_1, \ldots, i_l\rangle \in \Sigma$. In the special case $A = T_\Sigma\mathrm{V}$, we have

Proposition 7. *An assignment $\theta : Z \longrightarrow T_\Sigma V$ gives a structural valuation, and $\theta^* : M_\Sigma Z \longrightarrow T_\Sigma \mathrm{V}$ gives its "homomorphic" extension on meta-terms.*

To see why, first we note that the assignment θ is a family of maps $\theta(n) : Z(n) \longrightarrow T_\Sigma \mathrm{V}(n)$ such that

$$\theta(n) : \mathrm{z} \longmapsto t \in T_\Sigma \mathrm{V}(n).$$

Namely, it maps an n-ary metavariable z to some structural term t under n. Comparing the definition of structural valuation with this, and regarding the substitute $\underline{\lambda}(x_1, \ldots, x_n).t$ as $t \in T_\Sigma \mathrm{V}(n)$ (because θ is structural), both definitions coincide. Hence, hereafter we use the word "valuation" in this sense:

Definition 8. A *valuation* is an assignment $\theta : Z \longrightarrow T_\Sigma \mathrm{V}$ into the Σ-monoid of terms. Also, we use the following: a *meta-valuation* is an assignment $\theta : Z \longrightarrow M_\Sigma X$ into the Σ-monoid of meta-terms.

Now we know in what sense θ^* is a "homomorphic" extension of a valuation θ (which is not explained formally in the ordinary definitions [KOR93, OR94, DR98, Oos94]). Namely θ^* is a Σ-*monoid morphism*, which preserves Σ-algebra structure (i.e. Σ-homomorphism) and monoid structure.

4.4 Structural Valuations are Sufficient

A valuation in the original sense (Sect. 2) was a map $\theta : \mathrm{z} \mapsto \underline{\lambda}(x_1, \ldots, x_n).t$ where t is an *arbitrary* term, which means that $\underline{\lambda}(x_1, \ldots, x_n).t$ may have variables other than $x_1, \ldots, x_n$. But in the case of a structural valuation, variables in t are taken only from $x_1, \ldots, x_n$. We show that structural valuations are sufficient to generate CRS rewrite relation on terms if we make some weakening of rules.

For $m \leq m'$, let $\rho : \mathbb{N} \to \mathbb{N}$ be the function defined by $\rho(m+i) \triangleq m' + i$ for each $i \in \mathbb{Z}$. Suppose $\mathbb{N}$-indexed metavariable sets $Z' = Y \cup \{\mathrm{z}^m\}$, $Z = Y \cup \{\mathrm{z}^{m'}\}$, $\mathrm{z} \notin Y$. The *weakening of the arity of the metavariable* z *by* ρ *from* m *to* m' is a function ρ_z on (unstructural) meta-terms defined as follows.

$$\rho_\mathrm{z}(\mathrm{z}[t_1, \ldots, t_m]) = \mathrm{z}[1, \ldots, m'-m, \rho_\mathrm{z}(t_1), \ldots, \rho_\mathrm{z}(t_m)]$$
$$\rho_\mathrm{z}(n.t) = \rho(n).\rho_\mathrm{z}(t) \quad \rho_\mathrm{z}(F(\vec{t})) = F(\rho_\mathrm{z}(\vec{t})) \quad \rho_\mathrm{z}(x) = \rho(x) \qquad (x \in \mathbb{N}).$$

Notation 9. We may use the notation $Z|n \vdash s \to t$ for a rule or a rewrite step if metavariables and variables in s and t are included in Z and n respectively. We may also simply write $Z \vdash s \to t$ or $n \vdash s \to t$ if another part is not important.

Let $\mathcal{R}$ be a structural CRS that follows the method of de Bruijn levels. Then *weakening closure* of $\mathcal{R}$, denoted by $\mathcal{R}^\circ$, is defined by the following inference rules (i.e. the least set satisfying the rules):

$$\frac{l \to r \in \mathcal{R}}{l \to r \in \mathcal{R}^\circ} \qquad \frac{Y \cup \{\mathrm{z}^m\} \vdash l \to r \in \mathcal{R}^\circ}{Y \cup \{\mathrm{z}^{m+j}\} \vdash \vec{j}.\rho_\mathrm{z} l \to \vec{j}.\rho_\mathrm{z} r \in \mathcal{R}^\circ}$$

where ρ_{z} is weakening of the arity of the metavariable z from m to $m+j$ ($j \in \mathbb{N}$ is arbitrary). This means that although originally a metavariable z^m can be replaced with a term exactly containing m-variables, it will be weakened to z^{m+j}, which can be replaced with a term containing $m+j$-variables.

Then, we reformulate the generation of rewrite relation as follows:

$$\frac{Z \vdash \vec{n}.l \to \vec{n}.r \in \mathcal{R}}{n \vdash \theta^*(n)(l) \Rightarrow_{\mathcal{R}} \theta^*(n)(r)} \qquad \frac{n+i \vdash s \Rightarrow_{\mathcal{R}} t}{n \vdash F(\ldots, n+\vec{i}.s, \ldots) \Rightarrow_{\mathcal{R}} F(\ldots, n+\vec{i}.t, \ldots)}$$

where $\theta : Z : \longrightarrow T_\Sigma \mathrm{V}$ is a valuation.

Proposition 10. *For a structural CRS $\mathcal{R}$ that follows the method of de Bruijn levels, the ordinary definition (cf. Sect. 2) and the above definition with $\mathcal{R}^\circ$ generate the same rewrite relation on structural terms, i.e. $s \Rightarrow_{\mathcal{R}^\circ} t$ iff $s \to_{\mathcal{R}} t$ for structural terms s, t.*

5 Algebraic Semantics of Rewriting

In this section, we interpret rewrite rules of structural CRSs by Σ-binding algebras, and give a complete characterisation of termination in this framework. *Hereafter, in this paper we only consider structural CRSs.* So we just say "a CRS" for a structural CRS.

For a presheaf A, we write $>_A$ for a family of transitive relations $\{>_{A(n)}\}_{n\in\mathbb{N}}$, where $>_{A(n)}$ is a transitive relation on the set $A(n)$ for each $n \in \mathbb{N}$. In this paper, we use the following notion of monotonicity [Zan94].

Definition 11. Let $(A_1, >_{A_1}), \ldots, (A_l, >_{A_l}), (B, >_B)$ be presheaves equipped with transitive relations. A map $f : A_1 \times \cdots \times A_l \longrightarrow B$ in $\mathbf{Set}^{\mathbb{F}}$ is *monotone* if all $a_1, b_1 \in A_1(n), \ldots, a_l, b_l \in A_l(n)$ with $a_k >_{A(n)} b_k$ for some k and $a_j = b_j$ for all $j \neq k$, then $f(n)(a_1, \ldots, a_l) >_{B(n)} f(n)(b_1, \ldots, b_l)$.

We interpret rewrite rules in a $\mathrm{V}+\Sigma$-algebra.

Definition 12. Let A be a $\mathrm{V}+\Sigma$-algebra. A *term-generated assignment* $\phi : Z \longrightarrow A$ is a morphism of $\mathbf{Set}^{\mathbb{N}}$ that is expressed as the composite

$$Z \xrightarrow{\theta} T_\Sigma \mathrm{V} \xrightarrow{!_A} A$$

for some valuation θ, where $!_A$ is the unique $\mathrm{V}+\Sigma$-algebra homomorphism from the initial $\mathrm{V}+\Sigma$-algebra $T_\Sigma \mathrm{V}$. Throughout the paper, we denote by $!_A$ this unique $\mathrm{V}+\Sigma$-homomorphism.

This means that an interpretation of a metavariable z by a term-generated assignment θ is performed by firstly assigning to z some term t and then interpreting the term in a $\mathrm{V}+\Sigma$-algebra A. Why this is needed is that CRS rewrite relation is generated on terms (not on meta-terms). So, to interpret CRS rewrite rules, not all assignments are needed; only term-generated assignments are sufficient.

Definition 13. A *monotone* $\mathrm{V}+\Sigma$-*algebra* $(A, >_A)$ is a $\mathrm{V}+\Sigma$-algebra $A = (A, [\nu, [F_A]_{F\in\Sigma}])$, (where $\nu : \mathrm{V} \longrightarrow A$), equipped with a transitive relation $>_{A(n)}$ on $A(n)$ for each $n \in \mathbb{N}$ such that every operation f_A is monotone. Moreover, if $>_{A(n)}$ is a well-founded strict partial order for each $n \in \mathbb{N}$, A is called *well-founded*.

Definition 14. Let $\mathcal{R}$ be a CRS. A monotone V+Σ-algebra $(A, >_A)$ *satisfies* a CRS rewrite rule $Z \vdash \vec{n}.l \to \vec{n}.r$ if

$$\phi^*(n)(l) >_{A(n)} \phi^*(n)(r)$$

for all term-generated assignments $\phi : Z \longrightarrow A$. A $(\mathrm{V}+\Sigma, \mathcal{R})$-*algebra* A is a monotone V+Σ-algebra A that satisfies all rules in the weakening closure $\mathcal{R}^\circ$.

Define the $\mathbb{N}$-indexed transitive relation $\to^+_{\mathcal{R}(n)} \triangleq \{(s,t) \mid n \vdash s \Rightarrow^+_{\mathcal{R}^\circ} t\}$, where the latter $(-)^+$ denotes the transitive closure.

Theorem 15. *For a CRS* $\mathcal{R}$*,* $(T_\Sigma \mathrm{V}, \to^+_{\mathcal{R}})$ *is an initial* $(\mathrm{V}+\Sigma, \mathcal{R})$*-algebra, i.e. for any* $(\mathrm{V}+\Sigma, \mathcal{R})$*-algebra* A*, there exists a unique monotone homomorphism* $T_\Sigma \mathrm{V} \longrightarrow A$*.*

Proof. Let $(A, >_A)$ be a $(\mathrm{V}+\Sigma, \mathcal{R})$-algebra. Since $T_\Sigma\mathrm{V}$ is an initial V+Σ-algebra (Theorem 4), $!_A : T_\Sigma\mathrm{V} \longrightarrow A$ is a unique V+Σ-algebra homomorphism. So, the remaining task is to show $!_A$ is monotone. This is proved by induction on the structure of inference of $\Rightarrow_{\mathcal{R}}$ and induction on the length of $\Rightarrow^+$. Note that $!_A \circ \theta$ is term-generated and all operations F_A on A are monotone. □

The following states that $(\mathrm{V}+\Sigma, \mathcal{R})$-algebras are sound and complete for many-step rewrite relation (where Notation 9 is used).

Corollary 16. *Let* $\mathcal{R}$ *be a CRS. The followings are equivalent:*

(i) $n \vdash s \to^+_{\mathcal{R}} t$ *holds,*
(ii) $!_A(n)(s) >_{A(n)} !_A(n)(t)$ *for all* $(\mathrm{V}+\Sigma, \mathcal{R})$*-algebras* $(A, >_A)$*.*

Proof. (i)⇒(ii): By Theorem 15.
(ii)⇒(i): Take $(A, >_A) = (T_\Sigma\mathrm{V}, \to^+_{\mathcal{R}})$. □

Restricting the above corollary to the case of well-founded monotone algebras, we obtain a complete characterisation of terminating CRSs.

Theorem 17. *A CRS* $\mathcal{R}$ *is terminating if and only if there is a well-founded* $(\mathrm{V}+\Sigma, \mathcal{R})$*-algebra.*

Proof. (⇐): Let A be a well-founded $(\mathrm{V}+\Sigma, \mathcal{R})$-algebra. Assume $\mathcal{R}$ is non-terminating, i.e. there exists an infinite reduction sequence $n \vdash t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$. By Corollary 16, we have $!_A(n)(t_1) >_{A(n)} !_A(n)(t_2) >_{A(n)} \cdots$. This contradicts well-foundedness of $>_A$.
(⇒): When a CRS $\mathcal{R}$ is terminating, the initial $(\mathrm{V}+\Sigma, \mathcal{R})$-algebra $(T_\Sigma\mathrm{V}, \to^+_{\mathcal{R}})$ is a desired well-founded algebra, because the strict partial order $\to^+_{\mathcal{R}}$ is well-founded. □

Example 18 (Incompleteness of functional interpretation [Pol96]). Assume the metavariables $Z = \{\mathrm{F}^1, \mathrm{X}^1\}$ and the binding signature $\Sigma = \{c : \langle 0 \rangle\}$. Consider the CRS $\mathcal{R}$ consisting of the following only:

$$c(\mathrm{F}[\mathrm{F}[\mathrm{X}[1]]]) \to \mathrm{F}[\mathrm{X}[1]].$$

We want to show termination of $\mathcal{R}$. Intuitively, this termination seems easy to be proved because with any rewrite step the number of c-symbols decreases. Nevertheless the existing interpretation method of higher-order rewriting based on the model of hereditary monotone functionals *cannot show termination of* $\mathcal{R}$ due to incompleteness of the model [Pol94, Pol96]. In contrast to it, we *can show* termination of $\mathcal{R}$ by using Theorem 17 as follows. Take the monotone $\mathrm{V}+\Sigma$-algebra $(T_\Sigma \mathrm{V}, \succ_{T_\Sigma \mathrm{V}})$ where $s \succ_{T_\Sigma \mathrm{V}(n)} t$ iff the number of c-symbols in s and t decreases. Notice that now all terms in $T_\Sigma \mathrm{V}(n)$ are consisting of c and variables in n only. Hence, all assignments into $T_\Sigma \mathrm{V}$ are of the forms $\mathrm{F} \mapsto c^k(1)$, $\mathrm{X} \mapsto c^m(1)$ (k-times and m-times c's). This gives a well-founded $(\mathrm{V}+\Sigma, \mathcal{R})$-algebra $(T_\Sigma \mathrm{V}, \succ_{T_\Sigma \mathrm{V}})$, which implies termination of $\mathcal{R}$ by Theorem 17.

6 Algebraic Semantics of Meta-Rewriting

We go beyond the standard definition of rewriting of CRS, and consider rewriting on *meta-terms*, which we call *meta-rewriting*. In the literature, although meta-rewriting has not been formally defined, Oostrom considered the notions of meta-CR and meta-SN of CRS and pointed out each of them is not derived from CR and SN of CRS respectively ([Oos94] Sect. 3.4).

We consider *meta-termination*, i.e. termination of meta-rewriting. In this section, we give algebraic semantics of meta-rewriting. Basically we repeat the semantics in Sect. 5, but we use Σ-monoids instead of Σ-binding algebras for the semantics structure.

Rewriting on Meta-terms. First we formally define meta-rewriting. Let Z be an $\mathbb{N}$-indexed set of metavariables. For a CRS $\mathcal{R}$ in which any two rules have disjoint metavariables taken from Z (if not, rename rules suitably), we denote the CRS by $(\mathcal{R}, Z)$. We define the meta-rewriting relation $\rightsquigarrow_{\mathcal{R}}$ as follows:

$$\frac{\vec{n}.l \to \vec{n}.r \in \mathcal{R}}{n \vdash \theta^*(n)(l) \rightsquigarrow_{\mathcal{R}} \theta^*(n)(l)} \qquad \frac{n{+}i \vdash s \rightsquigarrow_{\mathcal{R}} t}{n \vdash F(\ldots, n{+}\vec{i}.s, \ldots) \rightsquigarrow_{\mathcal{R}} F(\ldots, n{+}\vec{i}.t, \ldots)}$$

$$\frac{\mathrm{Z} \in Z(l) \quad n \vdash s \rightsquigarrow_{\mathcal{R}} t}{n \vdash \mathrm{Z}[\ldots, s, \ldots] \rightsquigarrow_{\mathcal{R}} \mathrm{Z}[\ldots, t, \ldots]}$$

where θ is a *meta-valuation* $Z \longrightarrow M_\Sigma X$ (Definition 8). We say that $\mathcal{R}$ is meta-terminating if $\rightsquigarrow_{\mathcal{R}}$ is well-founded.

Definition 19. A *monotone* Σ*-monoid* $(A, >_A)$ is a Σ-monoid A equipped with a transitive relation $>_{A(n)}$ on $A(n)$ for each $n \in \mathbb{N}$ such that every operation is monotone. Moreover, if $>_{A(n)}$ is a well-founded strict partial order for each $n \in \mathbb{N}$, A is called *well-founded*.

Let $\mathcal{R}$ be a CRS. A monotone Σ-monoid $A = (A, >_A)$ *satisfies* a rewrite rule $Z \vdash \vec{n}.l \to \vec{n}.r \in \mathcal{R}$ if

$$\phi^*(n)(l) >_{A(n)} \phi^*(n)(r)$$

for all assignments[2] $\phi : Z \longrightarrow A$. If A satisfies all rules in the weakening closure $\mathcal{R}^\circ$, it is called $(\Sigma, \mathcal{R})$-*monoid*.

An important example of $(\Sigma, \mathcal{R})$-monoid is $(M_\Sigma Z, \rightsquigarrow^+_{\mathcal{R}})$.

Definition 20. Let (A, ν, β) be a monotone Σ-monoid, and $\phi : Z \longrightarrow A$ an assignment. Define the map $\sigma : Z \bullet A \longrightarrow A$ by the composite

$$Z \bullet A \xrightarrow{\phi \bullet \mathrm{id}_A} A \bullet A \xrightarrow{\beta} A.$$

The assignment ϕ is called *admissible* if σ is monotone[3].

Notice that the multiplication β need not to be monotone. Actually, it is rather difficult to find a Σ-monoid whose multiplication is monotone. The unit $\nu : \mathrm{V} \longrightarrow A$ is automatically monotone because V has no transitive relation.

The notion of admissible assignments is an important ingredient of interpretation of meta-rewriting. Arbitrary assignments are not suitable to interpret meta-rewriting because it may cause non-order preservation. For example, assume the constants $\Sigma = \{a, b, c\}$, the metavariable $Z = \{\mathrm{z}^1\}$ and the CRS $\mathcal{R} = \{a \to b\}$. Then we have a meta-rewriting $\mathrm{z}[a] \rightsquigarrow_{\mathcal{R}} \mathrm{z}[b]$. We interpret this rewrite step in the $(\Sigma, \mathcal{R})$-monoid $(M_\Sigma Z, \rightsquigarrow^+_{\mathcal{R}})$. Take the assignment $\phi : \mathrm{z} \longmapsto c$. Then, this does not preserve the order:

$$\phi^*(\mathrm{z}[a]) = c \not\rightsquigarrow_{\mathcal{R}} c = \phi^*(\mathrm{z}[b]).$$

We need "monotonic" interpretation of meta-rewriting to establish algebraic termination method. The idea of admissible assignment is motivated by to prohibit this kind of "non-monotonic" interpretation of a rewrite step.

This problem is already recognised by van de Pol [Pol94]. The notion of admissible assignments is analogue to his notion of strict functionals. Actually, we can show that hereditary monotone functionals in his model forms a monotone Σ-monoid and our admissible assignments into this monotone Σ-monoid is the same as the strict valuations at the second-order types. Hence, we can apply the method of termination proof using hereditary monotone functionals to CRSs. For instance, termination of the examples of higher-order rewrite systems given in [Pol94, Pol96] (and their CRS versions are in [Raa]) can be shown by using Σ-monoids of hereditary monotone functionals given in [Pol94, Pol96].

Now we show a theorem analogue to Theorem 15 stating $(M_\Sigma Z, \rightsquigarrow^+_{\mathcal{R}})$ is an "initial model". More precisely,

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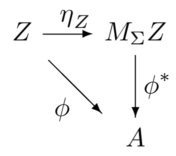
[2] Compare this definition with Definition 14 for rewriting.

[3] More precisely, $\sigma(n) : \coprod_{m \in \mathbb{N}} Z(m) \times A(n)^m / \sim \;\longrightarrow A(n)$ is monotone, i.e. if $\mathrm{z} \in Z(m)$ and all $a_1, b_1 \in A(n), \ldots, a_m, b_m \in A(n)$ with $a_k >_{A(n)} b_k$ for some k and $a_j = b_j$ for all $j \neq k$, we have $\sigma(n)(\mathrm{z}; a_1, \ldots, a_m) >_{A(n)} \sigma(n)(\mathrm{z}; b_1, \ldots, b_m)$.

Theorem 21.
For a CRS $(\mathcal{R}, Z)$*,* $(M_\Sigma Z, \rightsquigarrow^+_\mathcal{R})$ *is a free* $(\Sigma, \mathcal{R})$*-monoid over* Z*, i.e. for any admissible assignment* ϕ *from* Z *into a* $(\Sigma, \mathcal{R})$*-monoid* $(A, >_A)$*, there exists a unique* Σ*-monoid map* ϕ^* *that is monotone and makes the right diagram commute in* $\mathbf{Set}^\mathbb{F}$*, where* $\eta_Z : \mathrm{z}^l \longmapsto \mathrm{z}[1, \ldots, l]$*.*



Proof. Let $\phi : Z \longrightarrow A$ be an admissible assignment into a $(\Sigma, \mathcal{R})$-monoid $(A, >_A)$. Since $M_\Sigma Z$ is a free Σ-monoid [Ham04], ϕ^* is a unique Σ-monoid morphism that makes the above diagram commute in $\mathbf{Set}^\mathbb{F}$. So, the remaining task is to show ϕ^* is monotone. This is proved by induction on the structure of inference of $\rightsquigarrow_\mathcal{R}$ and the length of $\rightsquigarrow^+_\mathcal{R}$. The case for instantiation of a rewrite rule, we use $(\phi^* \circ \theta)^* = \phi^* \circ \theta^*$, which is proved by induction on meta-terms. The crucial case is to show ϕ^* preserves the relation of $\mathrm{z}[\ldots, s, \ldots] \rightsquigarrow_\mathcal{R} \mathrm{z}[\ldots, t, \ldots]$. This holds because we have assumed that ϕ is admissible. □

Theorem 22. *A CRS* $(\mathcal{R}, Z)$ *is meta-terminating if and only if there is a well-founded* $(\Sigma, \mathcal{R})$*-monoid.*

Proof. ($\Leftarrow$): Let A be a well-founded $(\Sigma, \mathcal{R})$-monoid. Assume $\mathcal{R}$ is not meta-terminating, i.e. there exists an infinite meta-rewriting sequence

$$Z|n \vdash t_1 \rightsquigarrow_\mathcal{R} t_2 \rightsquigarrow_\mathcal{R} t_3 \rightsquigarrow_\mathcal{R} \cdots .$$

By Theorem 21, for any admissible assignment $\phi : Z \longrightarrow A$, we have

$$\phi^*(n)(t_1) >_{A(n)} \phi^*(n)(t_2) >_{A(n)} \phi^*(n)(t_3) >_{A(n)} \cdots .$$

This contradicts well-foundedness of $>_A$.
($\Rightarrow$): When a CRS $\mathcal{R}$ is meta-terminating, the free $(\Sigma, \mathcal{R})$-monoid $(M_\Sigma Z, \rightsquigarrow^+_\mathcal{R})$ over Z is a desired well-founded one, because the strict partial order $\rightsquigarrow^+_\mathcal{R}$ is well-founded. □

7 Termination of Binding CRSs

Let $(\mathcal{R}, X)$ be a CRS such that every meta-application in rules of $\mathcal{R}$ is always of the form $\mathrm{z}^l[1, \ldots, l]$. We call such a CRS a *binding CRS* because it is essentially meta-application-free (cf. binding TRS [Ham03]). To interpret a rule and meta-rewriting in a binding CRS $\mathcal{R}$, we do not need the monoid structure of Σ-monoids, i.e. the multiplication β is not used. Because, for example, interpret the meta-term $\mathrm{z}[1, 2]$ (for z^2) in a rule by an assignment $\phi : X \longrightarrow A$ into a Σ-monoid (A, ν, β):

$$\phi^*(\mathrm{z}[1, 2]) = \beta(\phi(\mathrm{z}); \nu(1), \nu(2)) = \phi(\mathrm{z}).$$

This is due to $A \bullet \mathrm{V} \cong A$, i.e. V is the unit of the monoidal category $\mathbf{Set}^\mathbb{F}$. So, to interpret a meta-term like $\mathrm{z}[1, 2]$, we just need an assignment ϕ. Hence, we assume A to be a $X+\mathrm{V}+\Sigma$-algebra for interpretation of binding CRSs. Then we can replete the discussion of interpretation of meta-rewriting: A *satisfies* a rule $\vec{n}.l \to \vec{n}.r \in \mathcal{R}$ if $\phi^*(n)(l) >_A \phi^*(n)(r)$ for all assignments $\phi : X \longrightarrow A$ into $X+\mathrm{V}+\Sigma$-algebras. We denote by $B_\Sigma X$ an initial $X+\mathrm{V}+\Sigma$-algebra and call an element of it a *binding meta-term*. Notice that a binding CRS is a CRS built only from binding meta-terms. We define the meta-rewriting on binding meta-terms by $\to_{\mathcal{R}} \triangleq \rightsquigarrow_{\mathcal{R}} \cap \bigcup_{n\in\mathbb{N}}(B_\Sigma X \times B_\Sigma X)(n)$. Then, $(B_\Sigma X, \to^+_{\mathcal{R}})$ is an initial $(X+\mathrm{V}+\Sigma, \mathcal{R})$-algebra.

Proposition 23. *A binding CRS* $(\mathcal{R}, X)$ *is meta-terminating on binding meta-terms if and only if there is a well-founded* $(X+\mathrm{V}+\Sigma, \mathcal{R})$*-algebra.*

For a binding CRS $\mathcal{R}$, it is clear that meta-termination of $\mathcal{R}$ on binding meta-terms implies termination of $\mathcal{R}$ on terms because all terms are binding meta-terms (meta-application-free). Hence, in the case of binding CRSs this becomes an interesting termination proof method by interpretation because we do not need a monoid structure.

Example 24. We show termination of the CRS $\mathcal{R}$ for conversion into prenex normal form in the introduction. Formally, $\mathcal{R}$ is built from the binding signature $\Sigma = \{\forall, \exists : \langle 1\rangle, \wedge, \vee : \langle 0, 0\rangle, \neg : \langle 0\rangle\}$ and the metavariables $X = \{\mathrm{P}^0, \mathrm{Q}^1\}$. The structural CRS $\mathcal{R}$ in de Bruijn levels is obtained by just replacing the variable x with 1.

$$\mathrm{P} \wedge \forall(1.\mathrm{Q}[1]) \to \forall(1.\mathrm{P} \wedge \mathrm{Q}[1]) \qquad \neg\forall(1.\mathrm{Q}[1]) \to \exists(1.\neg(\mathrm{Q}[1]))$$
$$\forall(1.\mathrm{Q}[1]) \wedge \mathrm{P} \to \forall(1.\mathrm{P} \wedge \mathrm{Q}[1]) \qquad \neg\exists(1.\mathrm{Q}[1]) \to \forall(1.\neg(\mathrm{Q}[1])).$$

We use Proposition 23 to show termination. Take the $X+\mathrm{V}+\Sigma$-algebra K by $K(n) = \mathbb{N}$ with $>_{K(n)}$ by the usual order $>$ on $\mathbb{N}$ for all $n \in \mathbb{N}$. The operations are given by

$$\wedge_{K(n)}(x, y) = \vee_{K(n)}(x, y) = 2x + 2y$$
$$\neg_{K(n)}(x) = 2x \qquad \forall_{K(n)}(x) = \exists_{K(n)}(x) = x + 1.$$

All operations are monotone. We can show that K satisfies the rules: take an assignment $\phi : X \longrightarrow K$ by $\mathrm{P} \mapsto x \in \mathbb{N}$ and $\mathrm{Q} \mapsto y \in \mathbb{N}$, then

$$\phi^*(0)(\mathrm{P} \wedge \forall(1.\mathrm{Q}[1])) = 2x + 2(y + 1) >_{K(0)} (2x + 2y) + 1 = \phi^*(0)(\forall(1.\mathrm{P} \wedge \mathrm{Q}[1]))$$
$$\phi^*(0)(\neg\exists(1.\mathrm{Q}[1])) = 2(y + 1) >_{K(0)} 2y + 1 = \phi^*(0)(\forall(1.\neg(\mathrm{Q}[1]))).$$

Similar for other rules. Since $>_{K(0)}$ is well-founded, this shows K with ϕ is a well-founded $(X+\mathrm{V}+\Sigma, \mathcal{R})$-algebra. Thus, the binding CRS $\mathcal{R}$ is meta-terminating on binding meta-terms by Proposition 23. Hence $\mathcal{R}$ is terminating on all CRS terms. This interpretation is simpler than the hereditary monotone functional model given in [Pol96].

Example 25. The CRS $\mathcal{S}$ for a CPS translation in Example 3 is also shown to be terminating by the following polynomial interpretation: take the $X+\mathrm{V}+\Sigma$-algebra K by $K(n) = \mathbb{N}$ with the unit $\nu : \mathrm{V} \to K$, $i \mapsto 0$ and the operations:

$$\mathsf{CPS}_{K(n)}(e) = 5e + 5 \quad (\![e]\!)_{K(n)} = 5e + 1 \quad \overline{\lambda}_{K(n)}(e) = e \quad \lambda_{K(n)}(e) = e + 1$$
$$(e_0 \,\overline{\ }\, e_1)_{K(n)} = e_0 + e_1 \quad (e_0\ e_1)_{K(n)} = e_0 + e_1 + 1.$$

Checking this satisfies $\mathcal{S}$ is just by calculation. Hence $\mathcal{S}$ is terminating.

Namely, if a CRS is a binding CRS, we do not need functionals to interpret higher-order function symbols such as $\forall, \exists, \overline{\lambda}, \lambda$.

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Appendix: Elementary Description of The Category $\mathbf{Set}^{\mathbb{F}}$

For those who are not familiar with category theory, we devote this section to an elementary description of the central categorical structure used in this paper: the category $\mathbf{Set}^{\mathbb{F}}$ and related morphisms. The functor category $\mathbf{Set}^{\mathbb{F}}$ plays an central role in this paper. The objects of it are functors $\mathbb{F} \to \mathbf{Set}$ and the arrows are natural transformations between them. In more elementary term, an objects A of $\mathbf{Set}^{\mathbb{F}}$ (often written as $A \in \mathbf{Set}^{\mathbb{F}}$) is given by a $\mathbb{N}$-indexed set $\{A(n)\}_{n\in\mathbb{N}}$ with "the arrow part" i.e. for each function $\rho : m \to n \in \mathbb{F}$, we also need to give a function $A(\rho) : A(m) \longrightarrow A(n)$.

An arrow (or called a map, morphism) between objects $A, B \in \mathbf{Set}^{\mathbb{F}}$ is a natural transformation $f : A \longrightarrow B$; more elementary, it is given by a family of functions of the form $f(n) : A(n) \longrightarrow B(n)$ parameterised by all $n \in \mathbb{N}$ that satisfies the condition $\forall a \in A(m) \;.\; B(\rho)(f(m)(a)) = f(n)(A(\rho)(a))$ for all functions $\rho : m \to n$. This condition ("naturality") diagrammatically means the commutativity of the diagram

$$\begin{array}{ccccc} m & \quad & A(m) & \xrightarrow{f(m)} & B(m) \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle A(\rho)}\downarrow & & \downarrow{\scriptstyle B(\rho)} \\ n & & A(n) & \xrightarrow{f(n)} & B(n) \end{array}$$

Very roughly, we can think of $A \in \mathbf{Set}^{\mathbb{F}}$ as an $\mathbb{N}$-indexed set equipped with "something", and a map $f : A \longrightarrow B$ of $\mathbf{Set}^{\mathbb{F}}$ as an $\mathbb{N}$-indexed function $f(n) : A(n) \longrightarrow B(n)$ with "some coherence law". These "something" precisely mean the above descriptions. We may ignore them to get a rough understanding (with keeping in mind that these have officially such conditions). An object $A \in \mathbf{Set}^{\mathbb{F}}$ is often called a *presheaf*.