What is the Category for Haskell?

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IPL, 2007, September

We roughly believe · · ·

- ▷ not just **Set**
- ▷ maybe **CPO**, due to lazyness, and mixed variance ($D \cong D \Rightarrow D$)
- ▷ Folklore: initial algebra = finial coalgebra
  e.g. List datatype in Haskell is given by an initial algebra but contains all infinite lists
- ▷ But: there are subtle categorical differences in several variations of cpos

▶ Clarify these!

We clarify · · ·

- why coincidence?
  - which is a suitable category of cpos?
- why infinite data?
- how can we give a datatype functor?
  - a non-strict continuous function given by a strict function
- what is Haskell's algebraic datatype?

Categories of CPOs

Def.

A cpo is a poset such that every ω-chain has a lub.

NOTE: it may not have $\bot$.

Consider slightly different categories: **Cpo**, **Cppo**, $\mathbf{Cppo}_\bot$

All have cpos as objects.

Category	**Cpo**	**Cppo**	$\mathbf{Cppo}_\perp$
bottom?	no	$\perp$	$\perp$
functions	cont.	cont.	strict
$D \times E$	$+$	$+$	$+$
$[D \longrightarrow E]$	$+$	$+$	$\surd$
$D + E$	$+$	no	
Initial object	$\varnothing$	no	$\{\perp\}$
Terminal object	$\{*\}$	$\{\perp\}$	$\{\perp\}$
$D \otimes E$		$\surd$	$\surd$
$[D \longrightarrow_\perp E]$		$\surd$	$+$
$D \oplus E$		$\surd$	$+$
closed structure	ccc	ccc	monoidal closed
(co)completeness	$\surd$	no	$\surd$
i.alg.=final coalg.			$\surd$

General Construction of Initial Algebras

Thm. (Basic Lemma [Smyth-Plotkin])
Let $\mathcal{C}$ be a category with initial object $\mathbf{0}$, and $F : \mathcal{C} \longrightarrow \mathcal{C}$ a functor.
If F preserves the colimit of the ω-chain D

$$\mathbf{0} \xrightarrow{\ !\ } F(\mathbf{0}) \xrightarrow{\ F!\ } F^2(\mathbf{0}) \xrightarrow{\ F^2!\ } \cdots$$

then, the initial F-algebra exists and is given by the mediating arrow from the colimit of the ω-chain $F(\mathbf{0}) \longrightarrow F^2(\mathbf{0}) \longrightarrow \cdots$

$$\mathbf{colim}\ FD \ \cong\ F(\mathbf{colim}\ D) \rightarrow \mathbf{colim}\ D$$

▶ Why cocompleteness? At least, if $\mathcal{C}$ is $\cdots$

cocomplete (existence of any colimit)
$\Rightarrow \quad \mathbf{0} \longrightarrow F(\mathbf{0}) \longrightarrow F^2(\mathbf{0}) \longrightarrow \cdots$ has a colimit

Why Cocompleteness?

Cor.

Let $\mathcal{C}$ be a cocomplete category. If F is ω-cocontinuos functor, the initial F-algebra exists.

Note: If $\mathcal{C}$ is a cpo, this is Knaster-Tarski fixpoint theorem.

Construction of Colimits

Thm. [Mac Lane V.2.2]

Let $\mathcal{C}$ be a category with coproducts and coequalizers,
and $D : \mathbb{I} \longrightarrow \mathcal{C}$ a diagram in $\mathcal{C}$.
Then, $\mathbf{colim}\ D$ exists and is given by the coequalizer

$$\coprod_{(f:j\to k)\in \mathsf{arr}\mathbb{I}} D(j) \underset{[\iota_f]}{\overset{[\phi_f]}{\rightrightarrows}} \coprod_{i\in\mathbb{I}} D(i) \longrightarrow \mathbf{colim}\ D$$

where

$$\phi_f : D(j) \xrightarrow{D(f)} D(k) \xrightarrow{\iota} \coprod_i D(i)$$

$$\iota_f : D(j) \xrightarrow{\iota} \coprod_i D(i)$$

Example: List Types in $\mathbf{Cppo}_\perp$

Remark

Non-strict functions can be represented by strict functions in $\mathbf{Cppo}_\perp$

$$[\boldsymbol{X} \to \boldsymbol{Y}] \;\cong\; (\boldsymbol{X})_\perp \to_\perp \boldsymbol{Y}$$

Type constructors are $\times \; \otimes \; \oplus \; (-)_\perp$

In **Set**, the list functor is

$$\boldsymbol{F}\boldsymbol{X} = \mathbf{1} + \mathbb{N} \times \boldsymbol{X}$$

In $\mathbf{Cppo}_\perp$, assume $\mathbf{1}, \mathbb{N}$ to be flat cpos.

$\boldsymbol{F}\boldsymbol{X}$	cons :	how strict	i. alg.
$\mathbf{1} \oplus \mathbb{N} \times \boldsymbol{X}$	$\mathbb{N} \times \boldsymbol{L} \longrightarrow_\perp \boldsymbol{L}$	$\mathsf{cons}(\perp, \perp) = \perp$ but $\mathsf{cons}(\boldsymbol{a}, \perp) = \boldsymbol{a}{:}\perp$	fin.& inf.
$\mathbf{1} \oplus \mathbb{N} \otimes \boldsymbol{X}$	$\mathbb{N} \otimes \boldsymbol{L} \longrightarrow_\perp \boldsymbol{L}$	$\mathsf{cons}(\perp, \boldsymbol{l}) = \mathsf{cons}(\boldsymbol{a}, \perp) = \perp$	$\mathbb{N}^*$
$\mathbf{1} \oplus \mathbb{N} \otimes \boldsymbol{X}_\perp$	$\mathbb{N} \otimes \boldsymbol{L}_\perp \longrightarrow_\perp \boldsymbol{L}$	$\mathsf{cons}(\perp, \boldsymbol{l}) = \perp$ but $\mathsf{cons}(\boldsymbol{a}, \perp) = \boldsymbol{a}{:}\perp$	fin.& inf.
$\mathbf{1} \oplus \mathbb{N}_\perp \otimes \boldsymbol{X}_\perp$	$\mathbb{N}_\perp \otimes \boldsymbol{L}_\perp \longrightarrow_\perp \boldsymbol{L}$	$\mathsf{cons}(\perp, \perp) = \perp{:}\perp \neq \perp$	fin.& inf.

Remark

(1) $(\perp, \perp)$ is the least element of $\mathbb{N} \times \boldsymbol{L}$

(2) cons is a bistrict function

(3) cons is a left strict function

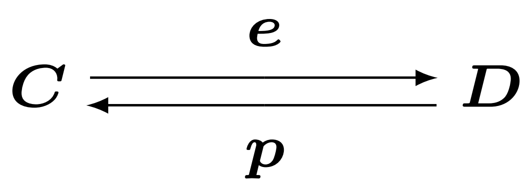
(4) $\mathbb{N}_\perp \otimes \boldsymbol{X}_\perp \cong (\mathbb{N} \times \boldsymbol{X})_\perp$, cons is continuous.
"continuous algebra". Haskell's datatype

Others

- Initial algebra construction via repeated application of $\boldsymbol{F}$
- Note: in $\mathbf{Cppo}_\perp$, infinite coproduct should contain lub of $\boldsymbol{\omega}$-chain while finite coproduct is just a crushed disjoint sum

Why Initial Algebra = Final Coalgebra?

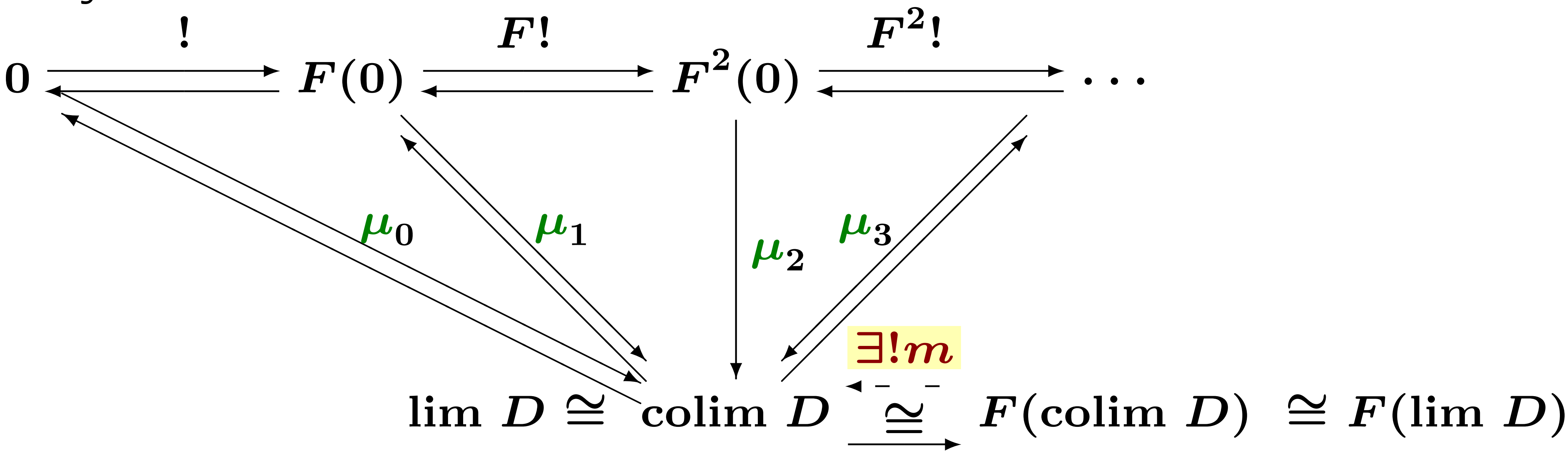
Def. Injection-Projection pair (e, p)



s.t. $p \circ e = \mathrm{id}, \quad e \circ p \sqsubseteq \mathrm{id}$.

If such a pair (e, p) exists, p is uniquely determined by e.

Why coincidence:



Actually, $\mu_i = F^i!$ and injective, so has projection

Datatypes and Cocomplete Categories

- ▷ **Set** for polynomial types

- ▷ $\mathbf{Cppo}_\perp$ for initial/finial (co)algebra coincidence & mixed variant types (“fold” given by Johan Glimming [CALCO’07] for $\mathcal{C}$ that is algebraically compact, has products, coproducts, smc, generator)

- ▷ **Cpo** for polynomial with monad types $\boldsymbol{T}$ (“fold” given by Filinski & Stovring [ICFP’07])

- ▷ $[\mathcal{C},\mathcal{C}]$ for nested datatypes, where $\mathcal{C}$ is ω-cocomplete (“generalised fold” given by Johann & Ghani [TLCA’07])

- ▷ $\mathbf{Set}^S$ for simple GADT (e.g. phantom type of Expr)

Conclusion: $\mathbf{Cppo}_\perp$ for core Haskell

Further Complication: Incorporating Function Types

To treat mixed-variant functor, e.g.,

$$FX = (X \to_{\perp} X)_{\perp}$$

local continuity is easier to chk than cocontinuity of datatype functor.

This means that we need to use order-enriched category $\mathbf{Cppo}_{\perp}$.

Thm. Every locally continuous functor

$$F : \mathbf{Cppo}_{\perp}^{op} \times \mathbf{Cppo}_{\perp} \to \mathbf{Cppo}_{\perp}^{op} \times \mathbf{Cppo}_{\perp}$$

has an initial algebra, which is also a final coalgebra.

References

(1) Smyth, Plotkin, *The category-theoretic solution of recursive domain equations*, SIAM J. Comput. Vol.11, No.4, 1982.

- ▷ Basic lemma (dual is for final coalgebra)
- ▷ Embedding-projection pair construction for mixed variance in enriched setting $\Rightarrow$ limit=colimit

(2) Lehmann, Smyth, *Algebraic Specification of Data Types: A Syntetic Approach*, Mathematical Systems Theory 14, 97-139, 1981.

- ▷ More detailed account for initial algebras for datatypes, such as lists, trees, stack, queues

(3) Abramsky, Jung, *Domain Theory*, Handbook of LiCS, Vol.3, 1994.

- ▷ Survey and modern account

(4) Fiore, PhD thesis, 1996.

(5) Simpson, Plotkin, Complete Axioms for Categorical Fixed-point Operators, LICS'00.