Algebraic Semantics of Higher-Order Abstract Syntax and Second-Order Rewriting

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This Lecture

I. Higher-Order Abstract Syntax

1. What is HOAS
2. Algebraic semantics
3. AIM: Meta-terms form a free $\boldsymbol{\Sigma}$-monoid

II. Second-Order Rewrite System

1. Definition of CRS
2. Algebraic interpretation
3. AIM: CRS rewriting forms a monotone $(\boldsymbol{\Sigma}, \boldsymbol{\mathcal{R}})$-algebra
4. Termination by interpretation

II. Algebraic Semantics of Second-Order Rewrite System

Algebraic Semantics of Second-Order Rewrite System

- ▷ Definition of CRS
- ▷ Monotone $\boldsymbol{\Sigma}$-algebra in $\mathbf{Set}^{\mathbb{F}}$
- ▷ Main Theorem
  A CRS $\boldsymbol{\mathcal{R}}$ is terminating iff there is a well-founded $(\mathbf{V}+\boldsymbol{\Sigma},\boldsymbol{\mathcal{R}})$-algebra.
- ▷ Examples

TRS: Review

Term Rewriting System (TRS) $\mathcal{R}$:

$$\begin{aligned} fact(0) &\rightarrow S(0) \\ fact(S(x)) &\rightarrow fact(x) * S(x) \end{aligned}$$

Terms are defined by

$$T_\Sigma(X) \ni \quad t \ ::= \ x \mid f(t_1, \ldots, t_m)$$

The rewrite relation

$$\rightarrow_{\mathcal{R}} \quad \subseteq \quad T_\Sigma(X) \times T_\Sigma(X)$$

is defined by

$$\frac{l \rightarrow r \in \mathcal{R}}{l\theta \rightarrow_{\mathcal{R}} r\theta} \qquad \frac{s \rightarrow_{\mathcal{R}} t}{f(\ldots, s, \ldots) \rightarrow_{\mathcal{R}} f(\ldots, t, \ldots)}$$

where θ is a substitution $\theta : X \rightarrow T_\Sigma(X)$

Combinatory Reduction System (CRS) [Klop'80]

Eg. A transformation to prenex normal forms

$$\mathrm{P} \wedge \forall(x.\mathrm{Q}[x]) \longrightarrow \forall(x.\mathrm{P} \wedge \mathrm{Q}[x]) \qquad \neg\forall(x.\mathrm{Q}[x]) \longrightarrow \exists(x.\neg(\mathrm{Q}[x]))$$

$$\forall(x.\mathrm{Q}[x]) \wedge \mathrm{P} \longrightarrow \forall(x.\mathrm{Q}[x] \wedge \mathrm{P}) \qquad \neg\exists(x.\mathrm{Q}[x]) \longrightarrow \forall(x.\neg(\mathrm{Q}[x]))$$

Def.

Variables	$x, y, z, \cdots$
Meta-variables	$\mathrm{Z}^{(l)}$ (arity l), $\cdots \in Z$
Function symbols	$f^{(l)}$ (arity l), $\cdots \in \Sigma$
Meta-terms	$t ::= x \mid f^{(l)}(\overrightarrow{x}_1.s_1, \ldots, \overrightarrow{x}_l s_l) \mid \mathrm{Z}^{(l)}[t_1, \ldots, t_l]$
Rewrite rules $\mathcal{R}$	$t_1 \longrightarrow t_2$ (with some syntactic conditions)
Rewrite relation	$\longrightarrow_{\mathcal{R}}$ on terms $\mathbf{T}_\Sigma \mathbf{V}$

$$\frac{l \longrightarrow r \in \mathcal{R}}{\theta^\sharp(l) \longrightarrow_{\mathcal{R}} \theta^\sharp(r)} \qquad \frac{s \longrightarrow_{\mathcal{R}} t}{f(\ldots, \overrightarrow{x}.s, \ldots) \longrightarrow_{\mathcal{R}} f(\ldots, \overrightarrow{x}.t, \ldots)}$$

Valuation $\theta : Z \longrightarrow \mathbf{T}_\Sigma \mathbf{V}$ maps a metavariable to a term as $\mathrm{Z} \mapsto t$

Presheaf with transitive relation $(A, >_A)$

Def. A presheaf $A \in \mathbf{Set}^{\mathbb{F}}$ is equipped with a transitive relation $>_A$ if

(1) $>_A$ is a family $\{>_{A(n)}\}_{n \in \mathbb{F}}$,
where $>_{A(n)}$ is a transitive relation on $A(n)$ (family)

(2) for all $a, b \in A(m)$ and $\rho : m \longrightarrow n$ in $\mathbb{F}$,
if $a >_{A(m)} b$, then $A(\rho)(a) >_{A(n)} A(\rho)(b)$. (natural)

Def. $(A_1, >_{A_1}), \ldots, (A_l, >_{A_l}), (B, >_B)$

A arrow $f : A_1 \times \cdots \times A_l \longrightarrow B$ in $\mathbf{Set}^{\mathbb{F}}$ is monotone if

$$f(n)(a_1, \ldots, a_l) >_{B(n)} f(n)(b_1, \ldots, b_l)$$

$A_k(n) \ni \; a_k >_{A(n)} b_k \; \in A_k(n)$ for some k, and
$A_j(n) \ni \; a_j \geq_{A(n)} b_j \; \in A_j(n)$ for all $j \neq k$.

Term-generated assignment

a $\mathbf{V} + \boldsymbol{\Sigma}$-algebra $\boldsymbol{A}$

a valuation $\boldsymbol{\theta} : \boldsymbol{Z} \longrightarrow \mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V}$ $\cdots$ an arrow of $\mathbf{Set}^{\mathbb{F}}$

Def. A term-generated assignment $\hat{\boldsymbol{\theta}} : \boldsymbol{Z} \longrightarrow \boldsymbol{A}$ is given by

$$\boldsymbol{Z} \xrightarrow{\ \boldsymbol{\theta}\ } \mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V} \xrightarrow{\ !_{\boldsymbol{A}}\ } \boldsymbol{A}$$

where $!_{\boldsymbol{A}}$ is the unique $\mathbf{V} + \boldsymbol{\Sigma}$-algebra homomorphism from an initial algebra $\mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V}$.

Examples

- $\boldsymbol{A} = \mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V}$ (Terms)
- $\boldsymbol{A} = \mathrm{M}_{\boldsymbol{\Sigma}}\boldsymbol{Z}$ (Meta-terms)
- $\boldsymbol{A} = \mathbf{H}$ (for clones)

Monotone Algebra

Def. A monotone $\mathbf{V}+\boldsymbol{\Sigma}$-algebra $(\boldsymbol{A}, >_{\boldsymbol{A}})$ is a $\mathbf{V}+\boldsymbol{\Sigma}$-algebra

$$\boldsymbol{A} = (\boldsymbol{A}, \{\boldsymbol{\nu}, \boldsymbol{f}^{\boldsymbol{A}}\}_{\boldsymbol{f} \in \boldsymbol{\Sigma}})$$

equipped with a transitive relation $>_{\boldsymbol{A}}$ such that every operation $\boldsymbol{f}^{\boldsymbol{A}}$ is monotone.

$\boldsymbol{A}$ is well-founded if $>_{\boldsymbol{A}(\boldsymbol{n})}$ is well-founded for every $\boldsymbol{n}$.

Monotone Algebra

a CRS $\mathcal{R}$

Def. A monotone $\mathrm{V}+\Sigma$-algebra $(A, >_A)$ satisfies a rewrite rule

$$Z \mid n \vdash l \rightarrow r$$

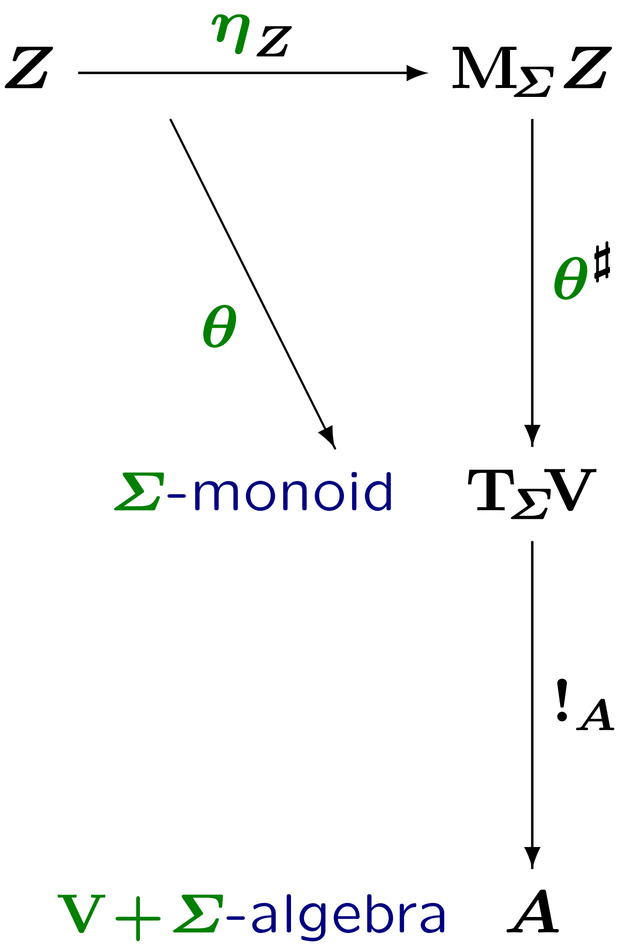
if for all term-generated assignments $\hat{\theta} : Z \rightarrow A$,

(1) $$\hat{\theta}_n^\sharp(l) \;>_{A(n)}\; \hat{\theta}_n^\sharp(r),$$

holds.

A $(\mathrm{V}+\Sigma, \mathcal{R})$-algebra A is
a monotone $\mathrm{V}+\Sigma$-algebra A that satisfies all rules in $\mathcal{R}$.

Interpretation in $(\mathbf{V}+\boldsymbol{\Sigma}, \boldsymbol{\mathcal{R}})$-algebra



$Z \mid n \vdash l \to r \in \mathcal{R}$ $\cdots$ a rule

a unique Σ-monoid morphism

$\theta^\sharp_n(l) \to_{\mathcal{R}} \theta^\sharp_n(r)$ $\cdots$ a rewrite

a unique $\mathrm{V}+\Sigma$-algebra homomor.

$!_A\theta^\sharp_n(l) >_{A(n+k)} !_A\theta^\sharp_n(l)$

$\cdots$ an interpretation

Example of $(\mathbf{V}+\boldsymbol{\Sigma}, \boldsymbol{\mathcal{R}})$-Algebra: Polynomial Interp.

▷ $\forall(\boldsymbol{y}.\exists(\boldsymbol{x}.(\neg\mathbf{p}(\boldsymbol{x}) \vee \mathbf{q}(\boldsymbol{y})))).$

▷ $\mathbf{V}+\boldsymbol{\Sigma}$-algebra $(\boldsymbol{K}_{\mathbb{N}}, >_{\boldsymbol{K}_{\mathbb{N}}})$ $\quad \boldsymbol{K}_{\mathbb{N}} \in \mathbf{Set}^{\mathbb{F}}$

– carrier $\boldsymbol{K}_{\mathbb{N}}(\boldsymbol{n}) = \mathbb{N}$

– operations $\quad \boldsymbol{\nu}_{\boldsymbol{n}}^{\boldsymbol{K}_{\mathbb{N}}}(\boldsymbol{x}) = \mathbf{0}$

$$\wedge_{\boldsymbol{n}}^{\boldsymbol{K}_{\mathbb{N}}}(\boldsymbol{x}, \boldsymbol{y}) = \vee_{\boldsymbol{n}}^{\boldsymbol{K}_{\mathbb{N}}}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{2}\boldsymbol{x} + \mathbf{2}\boldsymbol{y} \qquad \neg_{\boldsymbol{n}}^{\boldsymbol{K}_{\mathbb{N}}}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x}$$

$$\forall_{\boldsymbol{n}}^{\boldsymbol{K}_{\mathbb{N}}}(\boldsymbol{x}) = \exists_{\boldsymbol{n}}^{\boldsymbol{K}_{\mathbb{N}}}(\boldsymbol{x}) = \boldsymbol{x} + \mathbf{1}$$

All operations are monotone.

– $\boldsymbol{x} >_{\boldsymbol{K}_{\mathbb{N}}(\boldsymbol{n})} \boldsymbol{y} \quad \Leftrightarrow \quad \boldsymbol{x} >_{\mathbb{N}} \boldsymbol{y}$

▷ CRS

$$\mathrm{P}\wedge\forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \forall(\boldsymbol{x}.\mathrm{P}\wedge\mathrm{Q}[\boldsymbol{x}])$$

Important Example of $(\mathbf{V}+\boldsymbol{\Sigma},\boldsymbol{\mathcal{R}})$-Algebra

Terms with Rewrites

Prop. The presheaf $\mathbf{T_\Sigma V}$ of terms is equipped with the transitive relation $\{\longrightarrow^{+}_{\mathcal{R}(n)}\}_{n\in\mathbb{N}}$.

Proof. Transitivity of each $\longrightarrow^{+}_{\mathcal{R}(n)}$ is immediate.

It is also natural: we have

$$\frac{n \vdash s \longrightarrow_{\mathcal{R}} t}{n' \vdash \rho(s) \longrightarrow_{\mathcal{R}} \rho(t)}$$

for any $\rho : n \longrightarrow n'$ in $\mathbb{F}$, by induction on proof trees.

Important Example: Term Algebra

Thm. $(\mathbf{T_\Sigma V}, \to^+_{\mathcal{R}})$ is an $(\mathbf{V} + \boldsymbol{\Sigma}, \boldsymbol{\mathcal{R}})$-algebra.

Moreover, it is initial:
There exists a unique monotone homomorphism $\mathbf{T_\Sigma V} \longrightarrow \boldsymbol{A}$, for any $(\mathbf{V} + \boldsymbol{\Sigma}, \boldsymbol{\mathcal{R}})$-algebra $\boldsymbol{A}$.

Proof.

Since $\mathbf{T_\Sigma V}$ is an initial $\mathbf{V} + \boldsymbol{\Sigma}$-algebra, $!_{A} : \mathbf{T_\Sigma V} \longrightarrow \boldsymbol{A}$ is a unique $\mathbf{V} + \boldsymbol{\Sigma}$-algebra homomorphism.

It remains to show $!_A$ is monotone, i.e.

$$s \to^+_{\mathcal{R}} t \quad \Rightarrow \quad !_{A(n)}(s) \; >_{A(n)} \; !_{A(n)}(t)$$

By induction on the proof of $s \to^+_{\mathcal{R}} t$.

Corollary

Cor. $\quad n \vdash s \rightarrow^{+}_{\mathcal{R}} t \quad$ holds.

$$\Leftrightarrow$$

$$!_{A(n)}(s) \;>_{A(n)}\; !_{A(n)}(t)$$

for any monotone homomorphism $! : \mathbf{T}_{\Sigma}\mathbf{V} \longrightarrow A$ to $(\mathbf{V} + \Sigma, \mathcal{R})$-algebra$(A, >_A)$.

Proof. [⇒]: By the previous theorem.
[⇐]: Take $(A, >_A) = (\mathbf{T}_{\Sigma}\mathbf{V}, \rightarrow^{+}_{\mathcal{R}})$.

Main Theorem

Thm. A CRS $\mathcal{R}$ is terminating iff there is a well-founded $(\mathbf{V}+\boldsymbol{\Sigma}, \mathcal{R})$-algebra.

Proof. ($\Leftarrow$): Suppose a well-founded $(\mathbf{V}+\boldsymbol{\Sigma}, \mathcal{R})$-algebra $(A, >_A)$.

Assume $\mathcal{R}$ is non-terminating:

$$n \vdash t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots.$$

By the previous Corollary,

$$!_{A(n)}(t_1) >_{A(n)} !_{A(n)}(t_2) >_{A(n)} \cdots.$$

Contradiction.

($\Rightarrow$): When a CRS $\mathcal{R}$ is terminating, the initial $(\mathbf{V}+\boldsymbol{\Sigma}, \mathcal{R})$-algebra $(\mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V}, \to^{+}_{\mathcal{R}})$ is a desired well-founded algebra.

▶ Proof methond of termination of CRS:
Find a well-founded $(\mathbf{V}+\boldsymbol{\Sigma}, \mathcal{R})$-algebra.

Example

Binding signature $\boldsymbol{\Sigma} = \{\boldsymbol{c} : \langle \mathbf{0} \rangle\}$.   CRS $\boldsymbol{\mathcal{R}}$

$$\mathrm{F}^{\mathbf{1}}, \mathrm{X}^{\mathbf{1}} \mid \mathbf{1} \vdash \boldsymbol{c}(\mathrm{F}[\mathrm{F}[\mathrm{X}[\mathbf{1}]]]) \longrightarrow \mathrm{F}[\mathrm{X}[\mathbf{1}]].$$

- ▷ Intuitively, this CRS is terminating:
  at any rewrite step the number of $\boldsymbol{c}$-symbols decreases.
- ▷ The interpretation method of higher-order rewriting uses hereditary monotone functionals cannot show termination of $\boldsymbol{\mathcal{R}}$ due to the incompleteness [van de Pol '93 '96].
- ▷ Take the monotone $\mathbf{V}+\boldsymbol{\Sigma}$-algebra $(\mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V}, \succ_{\mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V}})$

$$\boldsymbol{s} \succ_{\mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V}(\boldsymbol{n})} \boldsymbol{t}$$

  if the number of $\boldsymbol{c}$-symbols in $\boldsymbol{s}$ and $\boldsymbol{t}$
- ▷ Any assignment into $\mathbf{T}_{\boldsymbol{\Sigma}}\mathbf{V}$ is of the form $\mathrm{F} \mapsto \boldsymbol{c}^{\boldsymbol{k}}(\boldsymbol{x}),\ \mathrm{X} \mapsto \boldsymbol{c}^{\boldsymbol{m}}(\boldsymbol{x})$
- ▷ This gives a well-founded $(\mathbf{V}+\boldsymbol{\Sigma}, \boldsymbol{\mathcal{R}})$-algebra.

Example

$$\mathbf{map}(x.\mathrm{F}[x], \mathbf{nil}) \longrightarrow \mathbf{nil}$$

$$\mathbf{map}(x.\mathrm{F}[x], \mathbf{cons}(y, ys)) \longrightarrow \mathbf{cons}(\mathrm{F}[y], \mathbf{map}(x.\mathrm{F}[x], ys))$$

▷ $\mathbf{V} + \boldsymbol{\Sigma}$-algebra, carrier: presheaf of clones $\mathbf{H} \in \mathbf{Set}^{\mathbb{F}}$

$$\mathbf{H}(0) = \mathbb{N}$$

$$\mathbf{H}(n) = (\mathbb{N}^n \longrightarrow \mathbb{N}) \quad (\text{for } n > 0)$$

$$\mathbf{H}(\rho)(f) = f \circ \langle \pi_{\rho 1}, \ldots, \pi_{\rho m} \rangle$$

operations (at n):

$$\mathbf{nil}^{\mathbf{H}}_n = \mathbf{K1} : \mathbb{N}^n \longrightarrow \mathbb{N}$$

$$\mathbf{cons}^{\mathbf{H}}_n(x, y) = (+) \circ \langle x, y, \mathbf{K2} \rangle : \mathbb{N}^n \longrightarrow \mathbb{N}$$

$$\mathbf{map}^{\mathbf{H}}_n(f, a) = f \circ \langle \mathrm{id}, \mathbf{K3} \rangle + (\times) \circ \langle a, f \circ \langle \mathrm{id}, a \rangle \rangle$$

$$\nu^{\mathbf{H}}_n(0) = \mathbf{K0}$$

Example of Termination Proof

metavariables $\mathrm{P}^{\mathbf{0}}$ and $\mathrm{Q}^{\mathbf{1}}$. CRS $\boldsymbol{\mathcal{R}}$.

$$
\begin{array}{llll}
\mathrm{P}\wedge\forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \forall(\boldsymbol{x}.\mathrm{P}\wedge\mathrm{Q}[\boldsymbol{x}]) & & \forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}])\wedge\mathrm{P} \longrightarrow \forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]\wedge\mathrm{P}) \\
\mathrm{P}\vee\forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \forall(\boldsymbol{x}.\mathrm{P}\vee\mathrm{Q}[\boldsymbol{x}]) & & \forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}])\vee\mathrm{P} \longrightarrow \forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]\vee\mathrm{P}) \\
\mathrm{P}\wedge\exists(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \exists(\boldsymbol{x}.\mathrm{P}\wedge\mathrm{Q}[\boldsymbol{x}]) & & \exists(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}])\wedge\mathrm{P} \longrightarrow \exists(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]\wedge\mathrm{P}) \\
\mathrm{P}\vee\exists(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \exists(\boldsymbol{x}.\mathrm{P}\vee\mathrm{Q}[\boldsymbol{x}]) & & \exists(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}])\vee\mathrm{P} \longrightarrow \exists(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]\vee\mathrm{P}) \\
\neg\forall(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \exists(\boldsymbol{x}.\neg(\mathrm{Q}[\boldsymbol{x}])) & & \neg\exists(\boldsymbol{x}.\mathrm{Q}[\boldsymbol{x}]) \longrightarrow \forall(\boldsymbol{x}.\neg(\mathrm{Q}[\boldsymbol{x}]))
\end{array}
$$

(1) Give the de Bruijn level notation version of $\boldsymbol{\mathcal{R}}$

(2) Show termination of $\boldsymbol{\mathcal{R}}$ by a polynomial interpretation.

Example of Termination Proof

▷ Just replace the variable x with 1.

$$\mathrm{P}\wedge\forall(1.\mathrm{Q}[1]) \rightarrow \forall(1.\mathrm{P}\wedge\mathrm{Q}[1]) \qquad \neg\forall(1.\mathrm{Q}[1]) \rightarrow \exists(1.\neg(\mathrm{Q}[1]))$$
$$\forall(1.\mathrm{Q}[1])\wedge\mathrm{P} \rightarrow \forall(1.\mathrm{P}\wedge\mathrm{Q}[1]) \qquad \neg\exists(1.\mathrm{Q}[1]) \rightarrow \forall(1.\neg(\mathrm{Q}[1])),$$

etc.

▷ Define a monotone $\mathbf{V}+\boldsymbol{\Sigma}$-algebra $(K, >_K)$

– carrier $K(n) = \mathbb{N}$

– order $>_{K(n)}$ is the standard order $>$ on $\mathbb{N}$

Take the operations as the polynomials.

$$\wedge^{K_\mathbb{N}}(x, y) = \vee^{K_\mathbb{N}}(x, y) = 2x + 2y$$
$$\neg^{K_\mathbb{N}}(x) = 2x \qquad \forall^{K_\mathbb{N}}(x) = \exists^{K_\mathbb{N}}(x) = x + 1.$$

All operations are monotone.

We show that $K_\mathbb{N}$ satisfies the rules: take an assignment

$\varphi : \boldsymbol{X} \longrightarrow \boldsymbol{K}_{\mathbb{N}}$ by $\mathrm{P} \mapsto \boldsymbol{x} \in \mathbb{N}$ and $\mathrm{Q} \mapsto \boldsymbol{y} \in \mathbb{N}$, then

$$\varphi^{\sharp}_{\boldsymbol{n}}(\mathrm{P} \wedge \forall(\mathbf{1}.\mathrm{Q}[\mathbf{1}])) = \mathbf{2}\boldsymbol{x} + \mathbf{2}(\boldsymbol{y} + \mathbf{1}) >_{\boldsymbol{K}_{\mathbb{N}}(\boldsymbol{n})} (\mathbf{2}\boldsymbol{x} + \mathbf{2}\boldsymbol{y}) + \mathbf{1}$$

$$= \varphi^{\sharp}_{\boldsymbol{n}}(\forall(\mathbf{1}.\mathrm{P} \wedge \mathrm{Q}[\mathbf{1}]))$$

$$\varphi^{\sharp}_{\boldsymbol{n}}(\neg\exists(\mathbf{1}.\mathrm{Q}[\mathbf{1}])) = \mathbf{2}(\boldsymbol{y} + \mathbf{1}) >_{\boldsymbol{K}_{\mathbb{N}}(\boldsymbol{n})} \mathbf{2}\boldsymbol{y} + \mathbf{1} = \varphi^{\sharp}_{\boldsymbol{n}}(\forall(\mathbf{1}.\neg(\mathrm{Q}[\mathbf{1}])))$$

Other rules are similar.

Since $>_{\boldsymbol{K}_{\mathbb{N}}(\boldsymbol{n})} = >$ is well-founded, this shows $\boldsymbol{K}_{\mathbb{N}}$ is a well-founded $(\mathbf{V} + \boldsymbol{\Sigma}, \boldsymbol{\mathcal{R}})$-algebra.

Hence the "binding" CRS $\boldsymbol{\mathcal{R}}$ is terminating.
(cf. Section 9 of the lecture note)