

Given a reduction system  $\rightarrow$  with term size  $| \cdot |$

(A):  $\exists$  binary relation  $\Rightarrow \subseteq \Lambda^2$  and translation  $* : \Lambda \rightarrow \Lambda$ :

- ① If  $M \rightarrow N$  then  $M \Rightarrow N$ .
- ② If  $M \Rightarrow N$  then  $M \twoheadrightarrow N$ .
- ③ If  $M \Rightarrow N$  then  $N \Rightarrow M^*$ .

(B):  $\exists$  monotonic functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$ :

If  $M \Rightarrow N$  then  $M \rightarrow^l N$  with  $l \leq g(|M|)$  and  $|N| \leq f(|M|)$ , where  $f$  and  $g$  are respectively in the  $p$ -th and  $q$ -th levels of the Grzegorzcyk hierarchy.

### Theorem (Quantitative Church–Rosser Theorem)

If  $M \xleftrightarrow[l]{r} N$  then there exists a term  $P$  such that  $M \rightarrow^m P^{k*}$  and  $N \rightarrow^n P^{k*}$  where  $k \leq \min\{l, r\}$ ,

- ①  $m \leq \sum_{i=0}^{r-1} g(f^{(i)}(|M|)), \quad n \leq \sum_{i=0}^{l-1} g(f^{(i)}(|N|)),$  and

- ②  $m, n$  are bounded by functions in the level of  $\max\{p + 1, q\} \geq 2$  of the Grzegorzcyk hierarchy. **23 / 56**