Some Improved Bounds on Communication Complexity via New Decomposition of Cliques

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Abstract

An ordered biclique partition of the complete graph \( K_n \) on \( n \) vertices is a collection of bicliques (i.e., complete bipartite graphs) such that (i) every edge of \( K_n \) is covered by at least one and at most two bicliques in the collection, and (ii) if an edge \( e \) is covered by two bicliques then each endpoint of \( e \) is in the first class in one of these bicliques and in the second class in other one. We show in this note that the minimum size of such a collection is \( O\left(\frac{n^2}{3}\right) \).

This gives new results on two problems related to communication complexity. Namely, (i) a new separation between the size of a fooling set and the rank of a 0/1-matrix, and (ii) an improved lower bound on the nondeterministic communication complexity of the clique vs. independent set problem are given.

keywords communication complexity, complete graphs, biclique partition, fooling set, rank

1 Introduction

Let \( G = (V, E) \) be an undirected graph. For \( U, W \subseteq V \) with \( U \cap W = \emptyset \), the complete bipartite graph with edge set \( U \times W \) is called a biclique and is denoted by \( B(U, W) \). A collection of bicliques \( \{B(U_i, W_i)\}_i \) is called a \( k \)-biclique covering of \( G \) if every edge in \( G \) lies in at least one and at most \( k \) bicliques in the collection. The minimum size of a \( k \)-biclique covering of \( G \) is denoted by \( bp_k(G) \). In particular, a 1-biclique covering is called a biclique partition and its minimum size \( bp_1(G) \) is just denoted by \( bp(G) \).

A famous theorem of Graham and Pollak [4] states that \( bp(K_n) = n - 1 \) where \( K_n \) denotes the complete graph on \( n \) vertices. Alon [1] generalized this to show

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that $\text{bp}_t(K_n) = \Theta(tn^{1/t})$. In this note, we consider an intermediate parameter between $\text{bp}(G)$ and $\text{bp}_2(G)$.

An ordered biclique covering of $G$ is a 2-biclique covering $\mathcal{B}(U_i, W_i)$ with an additional restriction that if an edge $e = \{u, v\}$ is covered by two bicliques, say $B(U_k, W_k)$ and $B(U_\ell, W_\ell)$, then each endpoint of $e$ belongs to distinct color class in these bicliques, i.e., $w \in U_k \cap W_\ell$ or $w \in U_\ell \cap W_k$ for $w \in \{u, v\}$. The minimum size of such a covering is denoted by $\text{bp}_{1.5}(G)$. Obviously from the definition, it holds that $\text{bp}_2(G) \leq \text{bp}_{1.5}(G) \leq \text{bp}(G)$.

In this note, we show that $\text{bp}_{1.5}(K_{n^3}) \leq (3n + 1)(n - 1)$ and hence $\text{bp}_{1.5}(K_n) = O(n^{2/3})$ by constructing an explicit such covering. This simple result is the main technical contribution of this note (Theorem 1 in Section 2.1).

One of the main motivations for considering such a parameter is its close connection to the problems on communication complexity which is one of the central topics in computational complexity. In particular, our covering gives new results on two problems related to communication complexity.

The first one is on the “rank” vs. “fooling set” problem. Let $M$ be an $m \times n$ 0/1-matrix. The rank of $M$ over the reals is denoted by $\text{rank}(M)$. A set $S \subseteq \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ of the index set of $M$ is called a fooling set for $M$ if there exists a value $z \in \{0, 1\}$ such that:

1. For every $(k, \ell) \in S$, $M_{k, \ell} = z$,

2. For any distinct $(k_1, \ell_1)$ and $(k_2, \ell_2)$ in $S$, $M_{k_1, \ell_2} \neq z$ or $M_{k_2, \ell_1} \neq z$.

The largest size of a fooling set of $M$ is denoted by $\text{fool}(M)$. Analyzing the size of a fooling set is one of the main tools for proving lower bounds on the communication complexity (see e.g., [7]).

It is known that $\text{fool}(M) \leq (\text{rank}(M) + 1)^2$ (see Dietzfelbinger, Hromkovič and Schnitger [2]). The open question is whether this quadratic gap can be improved or not (see e.g., [2, Open Problem 2]). Note that recently Friesen and Theis [3] proved that the exponent 2 on the rank is tight if we take the rank in a field of characteristic two.

For the original problem, the best separation between $\text{fool}(M)$ and $\text{rank}(M)$ known so far is $\text{fool}(M) \geq \text{rank}(M) \log_2 6$ ($\log_4 6 = 1.292 \cdots$), which is attributed to M.Hühne in [2]. In this note, we improve the exponent on the rank to $3/2$. In

\footnote{In [2] the value of the exponent $\log_4 6$ was described without giving an actual matrix. So we describe here what we believe to be their construction: Let $k \geq 2$ be an integer. Let $a_i$ ($1 \leq i \leq 2k$) be a 0/1-valued column vector of length $2k$, in which $k$ consecutive entries (in a rotating manner) starting from $i$-th entry have value 1. Let $A$ be a $2k \times 2k$ 0/1-matrix whose $i$-th column is $a_i$. Note that $\text{rank}(A) = k + 1$ and $\text{fool}(A) = 2k$. The $\ell$-th ($\ell \geq 1$) tensor $A^{\otimes \ell}$ of $A$ satisfies $\text{fool}(A^{\otimes \ell}) = \text{rank}(A^{\otimes \ell}) \log(2k)/\log(k+1)$. The exponent is maximized when $k = 3$.}
fact, our biclique covering immediately gives an infinite family of 0/1-matrices $M$ such that $\text{fool}(M) \geq \Omega(\text{rank}(M)^{3/2})$ (Theorem 2 in Section 2.2).

The second result derived from our covering is an improved lower bound on the nondeterministic communication complexity of the clique vs. independent set problem. The problem was introduced by Yannakakis [10], and is as follows: Consider a standard two-party communication game between Alice and Bob. Given a graph $G$, Alice gets as input a clique $C$ in $G$ and Bob gets as input an independent set $I$ in $G$. The goal of this problem, denoted by CL-IS$_G$ for short, is to output $|C \cap I|$, which is clearly 0 or 1. It is known that, for every graph $G$, the deterministic as well as the nondeterministic communication complexity for CL-IS$_G$ is between $\log_2 |V(G)|$ and $O(\log^2 |V(G)|)$ [10]. Getting an $\omega(\log |V(G)|)$ lower bound on the deterministic/nondeterministic communication complexity for this problem is considered as a big challenge (see e.g., [7, 8] and the references therein).

Recently, Huang and Sudakov [5] obtained the first nontrivial lower bound on the nondeterministic communication complexity $N^0(\text{CL-IS}_G) \geq (6/5) \log_2 |V(G)| - O(1)$ for an explicit family of graphs $G$. Their lower bound is made from combination of two separated results. The first one is to find a family of graphs $H$ with $\chi(H) = \Omega(\text{bp}(H)^{6/5})$, where $\chi(H)$ denotes the chromatic number of $H$, which gives a counterexample to the so-called Alon-Saks-Seymour conjecture in graph theory. The second one is a proposition due to Alon and Haviv (described in [5]) that gives a way to convert a gap between $\chi(H)$ and $\text{bp}(H)$ to a lower bound on the nondeterministic communication complexity on CL-IS$_G$ for a graph $G$ constructed from $H$.

It can be observed that an argument by Alon and Haviv works even if we replace $\text{bp}(H)$ with $\text{bp}_{1.5}(H)$. Since $\text{bp}_{1.5}(H)$ is smaller than $\text{bp}(H)$ in general, it would be easier to find a graph with a larger gap. In fact, we can see that to find a graph $H$ with $\chi(H) \geq f(\text{bp}_{1.5}(H))$ for some function $f(\cdot)$ is essentially equivalent to prove $\log_2 f(|V(G)|)$ lower bound on the nondeterministic communication complexity for CL-IS$_G$ for an explicit graph $G$. Remark that during the writing of this note, we learned that the proof of this equivalence (in a slightly different form) has been given by Lagoutte [9, Lemmas 21 and 22].

Our covering result can be stated as $\chi(H) = \Omega(\text{bp}_{1.5}(H)^{3/2})$ for $H = K_{n^3}$. By combining this with a modified argument of Alon and Haviv, we give an explicit family of graphs $G$ such that the nondeterministic communication complexity $N^0(\text{CL-IS}_G)$ is at least $(3/2) \log_2 |V(G)| - O(1)$ (Theorem 3 in Section 2.3).

All the results and their proofs are described in Section 2. Some experimental results on the parameter $\text{bp}_{1.5}(G)$ as well as some discussions are described in Section 3.
2 Results and Proofs

2.1 Ordered Partition of Complete Graphs

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. As defined in Introduction, for an undirected graph $G$, $bp_{1.5}(G)$ is the minimum size of an ordered biclique covering of $G$. The following is an example of such covering $\{B(U_i, W_i)\}_{i=1}^4$ of size four for $K_6$ on the vertex of $[6]$.

$$
U_1 = \{1, 2\}, \quad W_1 = \{4, 6\}, \\
U_2 = \{1, 3\}, \quad W_2 = \{2, 5\}, \\
U_3 = \{3, 6\}, \quad W_3 = \{1, 4\}, \\
U_4 = \{2, 4, 6\}, \quad W_4 = \{3, 5\}.
$$

The edges $\{1, 6\}$, $\{2, 3\}$ and $\{3, 4\}$ are covered twice. It can be checked that $(1, 6) \in (U_1 \times W_1) \cap (W_3 \times U_3)$, $(2, 3) \in (U_4 \times W_4) \cap (W_2 \times U_2)$ and $(3, 4) \in (U_3 \times W_3) \cap (W_4 \times U_4)$.

An easy case analysis verifies that $bp_{1.5}(K_6) = 4$. Note that $bp(K_6) = 5$ by Graham-Pollak Theorem and $bp_2(K_6) = 3$ by the witness

$$
U_1 = \{1, 2\}, \quad W_1 = \{3, 4, 5, 6\}, \\
U_2 = \{3, 4\}, \quad W_2 = \{5, 6\}, \\
U_3 = \{1, 3, 5\}, \quad W_3 = \{2, 4, 6\}.
$$

Hence these three parameters are all different for $K_6$. The values of these parameters for small complete graphs obtained by a computer search are shown in Section 3.

The rest of this section is devoted to show the following theorem.

**Theorem 1** $bp_{1.5}(K_{n^3}) \leq (3n + 1)(n - 1)$.

**Proof.** We first describe a construction of a collection of bicliques that gives a slightly weaker bound of $bp_{1.5}(K_{n^3}) \leq 4n(n - 1)$ since it is easier to explain. At the last of the proof, we slightly modify the construction to obtain the desired bound.

Consider the complete graph $K_{n^3}$ on the vertex set $V = [n]^3 = \{(x_1, x_2, x_3) | x_i \in [n]\}$. We define four subsets of the edge set of $K_{n^3}$:

$$
E_1 = \{\{u, v\} | u_1 = v_1 \text{ and } u_2 \neq v_2\}, \\
E_2 = \{\{u, v\} | u_2 = v_2 \text{ and } u_3 \neq v_3\}, \\
E_3 = \{\{u, v\} | u_3 = v_3 \text{ and } u_1 \neq v_1\}, \\
E_4 = \{\{u, v\} | u_1 \neq v_1 \text{ and } u_2 \neq v_2\}.
$$
It is easy to check that the union of \( E_i \)'s covers all edges in \( K_{n^3} \) on \([n]^3\), and \( E_i \) and \( E_j \) (\( i \neq j \)) are disjoint except for \( \{i, j\} = \{3, 4\} \). Namely,

\[
E_3 \cap E_4 = \{\{u, v\} \mid u_1 \neq v_1 \text{ and } u_2 \neq v_2 \text{ and } u_3 = v_3\}.
\]

Observe that each of \( E_1, E_2 \) and \( E_3 \) forms an \( n \)-blowup of \( n \) independent copies of \( K_n \), and \( E_4 \) is an \( n \)-blowup of the complement of \( n \times n \) grid graph. Let \( G_i \) be the “base” graph of \( E_i \), i.e., \( n \) independent copies of \( K_n \), and \( G_1 \) be the \( n \)-blowup of \( G_i \) so that \( E(G_1) = E_1 \). For \( i = 2, 3, 4 \), we similarly define \( G_i \) and \( G_i \). The vertices of \( G_i \) are labeled by three coordinates in \([n] \cup \{\ast\}\) in a natural way. For example, a vertex of \( G_3 \) is labeled by \((u_1, \ast, u_3)\) for \( u_1, u_3 \in [n] \). We will partition each \( G_i \) into \( n(n-1) \) bicliques.

Since the blowup of a biclique is a biclique itself, if we obtain a biclique partition of \( G_i \), then the blowup of all bicliques in this partition gives a biclique partition of \( G_i \) of the same size, i.e., \( \text{bp}(G_i) \leq \text{bp}(G_i) \). So we focus on the partition of base graphs \( G_i \) for a while.

Notice that, for every graph \( H \) on \( m \) vertices, say, \( \{v_1, \ldots, v_m\} \), the collection of \((m-1)\) stars \( \{B(\{v_i\}, N(v_i) \cap \{v_{i+1}, \ldots, v_m\})\}_{i=1}^{m-1} \) forms a biclique partition of \( H \), where \( N(v_i) \) denotes the set of neighbors of \( v_i \). In fact, for \( G_1, G_2 \) and \( G_3 \), \( n(n-1) \) stars are enough (instead of a trivial bound of \( n^2 - 1 \)) to cover all edges by noticing that we can skip the last vertices of each clique in the choice of a root of stars. Similarly, for \( G_4 \), we can skip the vertices in the last column if we arrange \( n^2 \) vertices into \( n \times n \) grid in a row-column order, and hence \( G_4 \) can be partitioned into \( n(n-1) \) stars.

In this fashion, we can obtain a collection of \( 4n(n-1) \) bicliques that covers all edge in \( K_{n^3} \). The point we should care is that every edge \( e = \{u, v\} \in E_3 \cap E_4 \) is covered by exactly two bicliques, say, \( B(U_k, W_k) \) and \( B(U_\ell, W_\ell) \), and so we must satisfy that \( w \in U_k \cap W_\ell \) or \( w \in U_\ell \cap W_k \) for \( w \in \{u, v\} \). This can easily be achieved by choosing the ordering of the roots of the stars carefully when we make the partitions of \( G_3 \) and \( G_4 \).

For \( G_3 \), we pick the roots of the stars according to the lexicographic order on the first and the third coordinates of a vertex, whereas for \( G_4 \) we pick them in the reverse of the lexicographic order on the first and the second coordinates. The key is that, for every edge \( \{u, v\} \in E_3 \cap E_4 \) of them are different, i.e., \( u_1 \neq v_1 \). In addition the blowups of \( G_3 \) and \( G_4 \) don’t “touch” the first coordinate of a label of a vertex. Hence in this way, we can guarantee that, for \( \{u, v\} \) with \( u_1 < v_1 \), \( u_1 \) is in the first class of a biclique in the collection for \( G_3 \) and in the second class of a biclique in the collection for \( G_4 \). This completes the proof of \( \text{bp}_{1.5}(K_{n^3}) \leq 4n(n-1) \).

The desired bound of \((3n+1)(n-1)\) is easily obtained by noticing that we can “merge” bicliques in the collections for \( E_1 \) and for \( E_4 \) that have a same partite set.
If we pick the roots of the stars according to the lexicographic order on the first and the second coordinates of a vertex when we make the partition of \( \tilde{G}_1 \), then the first class of a biclique in the collections for \( E_1 \) and for \( E_4 \) must be a member of

\[
\{(u_1, u_2, \ast) \mid (u_1, u_2) \in \mathbb{Z}_2 \backslash (1, n)\},
\]

which has the cardinality \( n^2 - 1 \). Two bicliques having a common first class can be merged into one bicliques, i.e.,

\[
B(U \times W_1) \cup B(U \times W_2) = B(U \times (W_1 \cup W_2)).
\]

Hence we can cover all edges in \( K_{n^3} \) by \( 2n(n - 1) + n^2 - 1 = (3n + 1)(n - 1) \) bicliques in total. This completes the proof of Theorem 1.

\[\square\]

### 2.2 Rank vs. Fooling Set

Given an ordered biclique partition \( \{B(U_i, W_i)\}_{i=1}^m \) (\( m := (3n + 1)(n - 1) \)) of \( K_{n^3} \) constructed in Theorem 1, let \( A_i \) (\( 1 \leq i \leq m \)) be an \( n^3 \times n^3 \) 0/1-matrix whose \((k, \ell)\)-entry is 1 iff \( k \in U_i \) and \( \ell \in W_i \). Let \( M = \sum_i A_i \), i.e., \( M \) is the component-wise sum of all \( A_i \)'s. Obviously, the rank of \( A_i \) is 1 for all \( i \), and hence the rank of \( M \) is at most \( m \).

Since \( \{B(U_i, W_i)\}_{i=1}^m \) forms an ordered biclique partition of \( K_{n^3} \), it is obvious that \( M \) is a 0/1-matrix and all the diagonal entries of \( M \) are zero. In addition, for every \( k \neq \ell \in [n]^3 \), at least one of \( M_{k, \ell} \) or \( M_{\ell, k} \) is one. This means that the set of all the diagonal entries of \( M \) forms a fooling set of \( M \). This immediately gives the following result on the gap between the rank and the fooling set.

**Theorem 2** For the matrix \( M \) defined above, fool\((M) = n^3 \) and \( \text{rank}(M) \leq 4n(n - 1) \). Thus, fool\((M) \geq \Omega((\text{rank}(M)^{1.5})) \). \[\square\]

More exactly speaking, we give a corresponding gap between the size of a 0-fooling set and the number of 1-monochromatic rectangles that cover all one entries in a 0/1-matrix.

### 2.3 Nondeterministic Communication Complexity for CL-IS

We use the standard notation of communication complexity (see e.g., [7]).

The nondeterministic communication complexity of a Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), denoted by \( N^1(f) \), is the smallest number of bits of communication needed to convince both Alice and Bob that \( f(x, y) = 1 \). Let \( M(f) \) be a \( 2^n \times 2^n \) 0/1-matrix whose \((x, y)\) entry is \( f(x, y) \), which is called a communication matrix of \( f \). The 1-cover number \( C^1(f) \) is the minimum number of monochromatic combinatorial rectangles needed to cover all 1 entries in \( M(f) \). It is known
that \( N^1(f) = \lceil \log C^1(f) \rceil \). \( N^0(f) \) and \( C^0(f) \) are defined similarly. We allow to write \( N^0(M) \) or \( C^0(M) \) instead of \( N^0(f) \) or \( C^0(f) \) when \( M = M(f) \) represents the communication matrix of \( f \).

For a graph \( G \), let \( M \) be a 0/1-matrix whose columns are indexed by cliques in \( G \) and rows are indexed by independent sets in \( G \), and for a clique \( C \) and an independent set \( I \), the value of \( (C, I) \)-entry of \( M \) is \(|C \cap I|\) (which is 0 or 1). The nondeterministic communication complexity \( N^1(\text{CL-IS}_G) \) and \( N^0(\text{CL-IS}_G) \) is defined by \( N^1(M) \) and \( N^0(M) \), respectively.

It is easy to check that \( N^1(\text{CL-IS}_G) \) is always \( \log_2 |V(G)| \), and so an interesting problem is to bound \( N^0(\text{CL-IS}_G) \). If one can find a graph \( H \) with \( \chi(H) = \Omega(\text{bp}(H)^\alpha) \) with \( \alpha > 1 \), then the following claim by Alon and Haviv (described in [5]) gives a construction of a graph \( G \) such that \( N^0(\text{CL-IS}_G) \geq \alpha \log_2 |V(G)| - O(1) \).

**Claim 1** Let \( H = (V, E) \) be a graph with vertex set \( H(G) = [n] \) such that \( \text{bp}(H) = m \). Then there exists a graph \( G \) with vertex set \([m]\) such that \( N^0(\text{CL-IS}_G) \geq \alpha \log_2 \chi(H) \).

By a quick inspection of the proof of Claim 1, we see that \( \text{bp}(H) \) in the claim can be replaced with \( \text{bp}_{1.5}(H) \) with no extra cost. The proof is analogous to the proof of Claim 1 [5, Section 4] which we describe here for completeness. Note that essentially the same argument has also been appeared in [9, Lemma 21].

**Claim 2** Let \( H = (V, E) \) be a graph with vertex set \( H(G) = [n] \) such that \( \text{bp}_{1.5}(H) = m \). Then there exists a graph \( G \) with vertex set \([m]\) such that \( N^0(\text{CL-IS}_G) \geq \log_2 \chi(H) \).

**Proof.** Let \( H \) be a graph with \( H(G) = [n] \) and \( \{B(U_i, W_i)\}_{i=1}^m \) be an ordered biclique partition of \( H \). For each biclique, we define the characteristic vector \( v_i = (v_{i1}, \ldots, v_{in}) \in \{0, 1,*\}^n \) so that

\[
v_{ij} = \begin{cases} 
0 & \text{if } j \in U_i, \\
1 & \text{if } j \in W_i, \\
* & \text{otherwise}.
\end{cases}
\]

We now define a graph \( G \) with vertex set \([m]\). Two vertex \( i \) and \( i' \) are adjacent in \( G \) if there exists a \( j \in [n] \) with \( v_{ij} = v_{i'j} = 1 \). Two vertex \( i \) and \( i' \) are nonadjacent in \( G \) if there exists a \( j \in [n] \) with \( v_{ij} = v_{i'j} = 0 \). For every other case, arbitrarily assign an edge or non-edge between \( i \) and \( i' \). Note that that \( G \) is well-defined since if there are distinct \( j, j' \in [n] \) such that \( v_{ij} = v_{i'j} = 0 \) and
\[ v_{ij'} = v_{i'j} = 1 \text{ then } j \in U_i \cap U_i' \text{ and } j' \in W_i \cap W_i' \] which contradicts to the assumption that \( \{B(U_i, W_i)\}_{i=1}^m \) is an ordered biclique partition.

Let consider the CL-IS problem on \( G \). Define \( C_j = \{ q \in [m] \mid v_{qj} = 1 \} \) and \( I_j = \{ q \in [m] \mid v_{qj} = 0 \} \). It is easy to check that \( \{C_j\} \) are cliques and \( \{I_j\} \) are independent sets in \( G \). Let \( M \) be the communication matrix of CL-IS\(_G\), and \( M' \) be the submatrix of \( M \) consisting of rows indexed by \( \{C_j\}_{j=1}^m \) and columns indexed by \( \{I_j\}_{j=1}^m \). Obviously, \( N^0(M) \geq N^0(M') \geq \log_2 C^0(M') \). We will show that \( C^0(M') \) is lower bounded by \( \chi(H) \).

Suppose that \( t \) \( 0 \)-monochromatic rectangles \( R_1, \ldots, R_t \) covers all diagonal entries of \( M' \). If \( M_{pp}' \) and \( M_{qq}' \) are both covered by \( R_i \), then \( M_{pq}' = M_{qp}' = 0 \) which implies \( C_p \cap I_q = C_q \cap I_p = \emptyset \). This means that \( (p, q) \) is not adjacent in \( H \). Hence, the collection of rectangles \( \{R_i\}_{i=1}^t \) naturally induces a coloring of vertices of \( H \) in \( t \) colors, and therefore \( C^0(M') \geq t \geq \chi(H) \).

By combining Theorem 1, Claim 2 and the trivial fact that \( \chi(K_n) = n \), we have the following theorem.

**Theorem 3** There exists an infinite family of graphs \( G \) such that \( N^0(\text{CL-IS}_G) \geq 1.5 \log_2 |V(G)| - O(1) \).

Before closing this section, we note that the converse of Claim 2 is also true.

**Claim 3** For every graph \( G \), there exists a graph \( H \) such that \( \text{bp}_{1.5}(H) \leq |V(G)| \) and \( \log_2 \chi(H) \geq N^0(\text{CL-IS}_G) \).

The proof of Claim 3 is again an analogous to the proof of the claim by Alon and Haviv (described in [5, Claim 4.1]) in which \( \text{bp}_{1.5}(H) \) is replaced with \( \text{bp}_2(H) \). So we only show a sketch of the proof here. Note that essentially the same argument has also been appeared in [9, Lemma 22].

Given a graph \( G \), we will construct a graph \( H \) from a communication matrix of CL-IS\(_G\) as follows. The vertices of \( H \) are all the pairs \( (C, I) \) such that \( C \) is a clique and \( I \) is an independent set in \( G \), and \( C \cap I = \emptyset \). Two vertices \( (C, I) \) and \( (C', I') \) are adjacent if \( C \cap I' \neq \emptyset \) or \( C' \cap I \neq \emptyset \).

For every vertex \( v_i \) in \( V(G) \), we define two subsets \( U_i = \{(C, I) \mid v_i \in C\} \) and \( W_i = \{(C, I) \mid v_i \in I\} \) of \( V(H) \). It is not hard to see that the collection \( \{B(U_i, W_i)\}_{i=1}^{|V(G)|} \) forms an ordered biclique partition of \( H \). In addition, we can check that \( \log_2 \chi(H) \geq \log_2 C^0(M) \geq N^0(\text{CL-IS}_G) \). See [5] for more detail.

In light of Claims 2 and 3, we see that the problem to find a graph \( H \) with \( \chi(H) \geq f(\text{bp}_{1.5}(H)) \) for a function \( f(\cdot) \) is essentially equivalent to prove \( \log_2 f(|V(G)|) \) lower bound on the nondeterministic communication complexity \( N^0(\text{CL-IS}_G) \) for an explicit graph \( G \). Since it is known that \( \text{bp}_{1.5}(K_n) \geq \text{bp}_2(K_n) =
Table 1: The values of $bp(K_n)$, $bp_{1.5}(K_n)$ and $bp_2(K_n)$.

<table>
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<th>3</th>
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<td>$bp_2(K_n)$</td>
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$\Theta(n^{1/2})$, we should consider non-complete graphs in order to beat $2\log_2 |V(G)|$ bound for CL-IS$G$, which is the best known lower bound on the deterministic communication complexity for CL-IS$G$ [6].

3 Concluding Remarks

We have shown that $bp_{1.5}(K_n) = O(n^{2/3})$, and it is known that

$$\Theta(n^{1/2}) = bp_2(G) \leq bp_{1.5}(G) \leq bp(G) = n - 1.$$ 

Apparently to determine the asymptotic behavior of $bp_{1.5}(K_n)$ is an interesting open question.

Table 1 shows the exact values of $bp(K_n)$, $bp_{1.5}(K_n)$ and $bp_2(K_n)$ for small values of $n$. The values for $bp_{1.5}(K_n)$ and $bp_2(K_n)$ are obtained by a computer search. We formulate the problem for finding a desired covering as an IP (integer programming) problem in an obvious way, and solve it using an IP solver. At the timing of writing this note, the values of $bp_{1.5}(K_n)$ are known for $n \leq 11$. The first open case is $bp_{1.5}(K_{12})$ which is 7 or 8. The experimental results suggest that $bp_{1.5}(K_n)$ would be truly intermediate between $bp(K_n)$ and $bp_2(K_n)$.

A final remark is that Table 1 verifies that $bp_2(K_n) = \lceil \sqrt{n} \rceil + \lfloor \sqrt{n} \rfloor - 2$ for $n \leq 16$. This matches to an upper bound given by Alon [1]. The best known lower bound on $bp_2(K_n)$ is $(1 + o(1))\sqrt{n}$ [5]. As was described in [5], to close the gap between the upper and lower bounds on $bp_2(K_n)$ would be interesting.

References


